ISEG - Lisbon School of Economics and Management

List of Exercises - Chapters 6 and 7
1^{st} Semester of 2020/2021

December 19, 2020

1. A box contains 10 balls, of which 3 are red, 2 are yellow, and 5 are blue. Five balls are randomly selected with replacement.

   (a) Calculate the probability that less than 2 of the selected balls are red.

   (b) Assume now that the five balls are randomly selected without replacement. Compute the previous probability.

2. Prove that if $X_1 \sim B(n_1, p)$ and $X_2 \sim B(n_2, p)$ and X_1 and X_2 are independent random variables, then $X_1 + X_2 \sim B(n_1 + n_2, p)$

   (**Hint:** Recall that if $X \sim B(n, p)$, then $M_X(t) = \left[(1-p) + pe^t\right]^n$.)

3. Let $X \sim B(n, p)$ and $X^* \sim B(n, 1-p)$, show that $P\left(n - X^* = x\right) = P\left(X = x\right).$

4. Just prior to jury selection for O. J. Simpson's murder trial in 1995, a poll found that about 20% of the adult population believed Simpson was innocent (after much of the physical evidence in the case had been revealed to the public). Ignore the fact that this 20% is an estimate based on a subsample from the population; for illustration, take it as the true percentage of people who thought Simpson was innocent prior to jury selection. Assume that the 12 jurors were selected randomly and independently from the population

   (a) Find the probability that the jury had at least one member who believed in Simpson's innocence.

   (b) Find the probability that the jury had at least two members who believed in Simpson's innocence.

   (c) What is the expected value and the variance of the number of jurors, in a sample of 12 jurors, who believed in Simpson's innocence?

5. A student takes a multiple choice test with 20 questions, each with 4 choices and only one is correct.
   (a) Assume that the student blindly guesses and gets one question correct. Find the probability that the reader has to read no more than 4 questions until we get the right one.
   (b) Assume that the student blindly guesses and gets two question correct. Find the probability that the reader has to read 10 questions until he gets the second question that is correct.
6. Past Experience indicates that an average number of 6 customers per hour stop for petrol at a petrol station. Assuming that the number of customers that stop for petrol at a petrol station is a Poisson random variable:
   (a) What is the probability of having 3 customers stopping in any hour?
   (b) What is the probability of having 3 customers or less stopping in any hour?
   (c) What is the expected value, and standard deviation of the distribution.
7. The average number of trucks arriving on any one day at a truck depot in a certain city is known to be 2.
   (a) If we assume a Poisson distribution, what is the probability that on a given day fewer than 2 trucks will arrive at this depot?
   (b) Assume that in a set of 10 random days the number of trucks arriving at the truck depot is independent in each day. Compute the probability of having fewer than 2 trucks arriving at the truck depot each day, for 5 days.
8. (Lack of memory of the exponential random variable) Let $X \sim Exp(\lambda)$, prove that $P(X > x + s | X > x) = P(X > s)$ for any $x \geq 0$ and $s \geq 0$.
9. Let $X_i \sim Exp(\lambda_i)$, $i = 1, 2$, be independent random variables. Prove that $Y = \min\{X_1, X_2\} \sim Exp(\lambda_1 + \lambda_2)$.

   (**Hint:** Note that $P(Y > y) = P(X_1 > y, X_2 > y)$).
10. Let $X_i \sim Exp(\lambda)$, $i = 1, 2$, be independent random variables.
    (a) Find the distribution of $Y_1 = \min\{X_1, X_2\}$ and $Y_2 = \max\{X_1, X_2\}$.
    (b) Find the expected value of $Z = \lambda^2 Y_1 - \frac{2}{3} Y_2$.
11. The lifetime in years of an electronic component is a continuous random variable X that follows

$$X \sim Exp(1)$$

(a) Find the lifetime L which a typical component is 60% certain to exceed.

(b) If five components are sold to a manufacturer, find the probability that at least one of them will have a lifetime less than L years.

12. The time intervals between successive trains stopping in a certain rail station have an exponential distribution with mean 6 minutes.

    (a) Find the probability that the time interval between two consecutive trains is less than 5 minutes.

    (b) Find a time interval t such that we can be 95% sure that the time interval between two successive trains will be greater than t.

    (c) Assume that the number of trains arriving in one hour is modeled by a Poisson random variable. Compute the probability that in a random hour 5 trains stop at the train station.

    (d) If we have counted 8 trains in the first hour, what is the probability that two of them arrived in the first 30 minutes?

13. Compute the following probabilities:

    (a) If Y is distributed $\chi^2(4)$ find $P(Y \leq 7.78)$.

    (b) If Y is distributed $\chi^2(10)$ find $P(Y > 18.31)$.

    (c) If Y is $\chi^2(1)$ find $P(Y \leq 3.8416)$.

14. Using the moment generating function, show that if $X \sim Gamma(a, b)$ and $Y = 2X/b$, then $Y \sim \chi^2(2a)$.

15. Prove that if X_1 and X_2 are independent random variables with Gamma distribution $X_1 \sim Gamma(a_1, b)$ and $X_2 \sim Gamma(a_2, b)$, then $X_1 + X_2 \sim Gamma(a_1 + a_2, b)$. (**Hint:** Recall that if $X \sim Gamma(a, b)$, then $M_X(t) = (1 - bt)^{-a}$ for $t < 1/b$).

16. Suppose customers arrive at a store according to a Poisson process, where the expected number of customers per hour is 0.5.

    (a) Knowing that 4 customers arrived at the store during the morning (4 hours) what is the probability that in this day (8 hours) the store receives more than 15 customers?

    (b) Compute the probability that the first customer does not arrive during the first hour (since the opening hour of the store).

    (c) What is the distribution of the time until the second customer arrives?

    (d) Find the probability that one has to wait at least half an hour until the second customer arrives.

(e) Find the probability that one has to wait at least five hours until the fourth customer arrives.

17. Compute the following probabilities:

    (a) If Y is distributed $N(1, 4)$, find $P(Y \leq 3)$.

    (b) If Y is distributed $N(3, 9)$, find $P(Y > 0)$.

    (c) If Y is distributed $N(50, 25)$, find $P(40 \leq Y \leq 52)$.

    (d) If Y is distributed $N(0, 1)$, find $P(|Y| > 1.96)$.

18. Prove that if the random variables $X_i, i = 1, 2$ have a normal distribution, $X_i \sim N(\mu_i, \sigma_i^2)$, and are independent and if $Y = aX_1 + bX_2 + c$, then $Y \sim N(\mu_Y, \sigma_Y^2)$, where $\mu_Y = a\mu_1 + b\mu_2 + c$ and $\sigma_Y^2 = a^2\sigma_1^2 + b^2\sigma_2^2$.

    (**Hint:** Recall that if $X \sim N(\mu, \sigma^2)$, then $M_X(t) = e^{\left(\mu t + 0.5\sigma^2 t^2\right)}$ and note that functions of independent random variables are also independent).

19. Suppose that diameter of a certain component produced in a factory can be modeled by a normal distribution with mean $10cm$ and standard deviation $3cm$.

    (a) Find the probability that the diameter of a random component is larger than $13cm$.

    (b) Find the probability that the diameter of a random component is less than $7cm$.

    (c) Selecting randomly 10 components, what is the probability that 2 of them have a diameter less than $7cm$?

    (d) What is the expected number of components that we have to inspect to find 1 with a diameter less than $7cm$?

20. A baker knows that the daily demand for a specific type of bread is a random variable X such that $X \sim N(\mu = 50, \sigma^2 = 25)$. Find the demand which has probability 1% of being exceeded.

21. Assume that X_i, with $i = 1, 2, 3$ represent the profit, in million of euros, of 3 different companies located in 3 different countries. If

$$X_1 \sim N(1, 0.01), \quad X_2 \sim N(1.5, 0.03), \quad X_3 \sim N(2, 0.06)$$

    (a) Which company is more likely to have a profit greater than 1.5 millions?

    (b) What is the probability of the profit of these 3 companies does not exceed 4 millions of euros? (Assume independence.)

22. The time elapsed since failure until repair (designated as repair time) of a certain type of machines is a random variable with exponential distribution with mean of 2 hours.

(a) What is the probability that a broken machine has a repair time of 1 hour or less?

(b) If 10 broken machines were randomly selected, what is the probability of the fastest repair be performed in less than 15 minutes? (assume independence)

(c) What is the probability that the total repair time of 50 broken machines does not exceed 90 hours? (assume independence)

23. Suppose that you roll a balanced die 36 times. Let Y denote the sum of the outcomes in each of the 36 rolls. Estimate the probability that $108 \leq Y \leq 144$.

24. Suppose that a book with 300 pages contains on average 1 misprint per page. Assume that the number of misprints per page is a Poisson random variable.

    (a) What is the probability that a random page has 2 or more misprints?

    (b) What is the probability that there will be at least 100 pages which contain 2 or more misprints? (assume independence)

    (c) What is the probability that there will be no more than 200 misprints in the book?

25. Assume that the number of hours per week that a student spends studying for the course of Statistics 1 follows a continuous uniform distribution in the interval $(0, 5)$.

    (a) What is the probability that a random student spends more than 3 hours studying for the course of Statistics 1?

    (b) In a group of 300 students, what is the probability that more than 100 spend more than 3 hours studying for the course of Statistics 1?

    (c) In a group of 300 students, what is the probability that, on average, students spend more than 4 hours studying for the course of Statistics 1?