ISEG - Lisbon School of Economics and Management

List of Exercises - Chapter 5
2^{nd} Semester of 2019/2020

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1. If X and Y have the joint probability distribution

$$f(x,y) = 1/4, \text{ for all } (x,y) \in \{(-3,-5),(-1,-1),(1,1),(3,5)\}.$$

Compute the $E(Z)$, where $Z = XY - 2X$.

Solution: By definition, $E(Z^2) = E((XY-2X)^2)$ is given by

$$E((XY-2X)^2) = \sum_{(x,y)\in D_{(X,Y)}} (xy-2x)^2 f_{X,Y}(x,y).$$

Since $f_{X,Y}(x,y) = 1/4$ in $(x,y) \in D_{(X,Y)}$, we have that

$$E((XY-2X)^2) = \frac{1}{4}\sum_{(x,y)\in D_{(X,Y)}} (xy-2x)^2 = 532/4 = 133.$$

2. Let X and Y be two random variables such that

$$f_{X,Y}(x,y) = \begin{cases} e^{-x-y}, & x>0, y>0 \\ 0, & otherwise \end{cases}$$

Compute $E(XY)$ and $E(X)$.

Solution: By definition $E(XY)$ is given by

$$\begin{aligned} E(XY) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy f_{X,Y}(x,y)dxdy = \int_{0}^{+\infty}\int_{0}^{+\infty} xye^{-x-y}dxdy \\ &= \int_{0}^{+\infty} ye^{-y}\int_{0}^{+\infty} xe^{-x}dxdy = 1. \end{aligned}$$

To calculate $E(X)$, one can notice that

$$E(X) = \int_{-\infty}^{+\infty} x f_X(x)dx = \int_{-\infty}^{+\infty} x \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dydx$$
$$= \int_{0}^{+\infty} xe^{-x} \int_{0}^{+\infty} e^{-y}dydx = 1.$$

3. If X and Y have the joint probability distribution

X/Y	-1	0	1
0	0	1/6	1/12
1	1/4	0	1/2

Prove that

(a) $cov(X,Y) = 0$;
Solution: Taking into account the properties of the covariance, we know that

$$Cov(X,Y) = E(XY) - E(X)E(Y).$$

therefore, we need to compute

$$E(XY) = \sum_{x=0}^{1}\sum_{Y=-1}^{1} xyP(X=x, Y=y) = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$$
$$E(Y) = \sum_{x=0}^{1}\sum_{y=-1}^{1} yP(X=x, Y=y) = \sum_{y=-1}^{1} yP(Y=y) = \frac{1}{3}$$
$$E(X) = \sum_{x=0}^{1}\sum_{x=-1}^{1} xP(X=x, Y=y) = \sum_{x=0}^{1} xP(X=x) = \frac{3}{4}$$

Therefore $Cov(X,Y) = \frac{1}{4} - \frac{3}{4} \times \frac{1}{3} = 0.$

(b) the two random variables are not independent. To check the random variables are dependent, one can check that

$$0 = P(X=0, Y=-1) \neq P(X=0) \times P(Y=-1) = \frac{1}{4} \times \frac{1}{4}.$$

4. If the probability density of X is given by

$$f(x) = \begin{cases} 1+x & ,\text{for } -1 < x \leq 0 \\ 1-x & ,\text{for } 0 < x < 1 \\ 0 & ,\text{elsewhere} \end{cases}$$

and $U = X$ and $V = X^2$, show that

(a) $cov(U,V)=0$;

(b) U and V are dependent.

5. Let X_1,X_2, and X_3 be independent random variables with means 4, 9, and 3 and the variances 3, 7, and 5.

   (a) Find the means and the variances of $Y = 2X_1 - 3X_2 + 4X_3$ and $Z = X_1 + 2X_2 - X_3$.
   Solution: $E(Y) = -7$, $Var(Y) = 155$, $E(Z) = 19$ and $Var(Z) = 36$.

   (b) Repeat (a) and (b) dropping the assumption of independence and using instead the information that $cov(X_1, X_2) = 1$, $cov(X_2, X_3) = -2$, and $cov(X_1, X_3) = -3$.
   Solution: $E(Y) = -7$, $Var(Y) = 143$, $E(Z) = 19$ and $Var(Z) = 54$.

6. If the joint probability density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{3}(y+x) & \text{, for } 0 < x \le 1,\ 0 < y < 2 \\ 0 & \text{, elsewhere} \end{cases}$$

Compute $Var(X|Y=y)$.

Solution: We start ba noticing that $Var(X|Y=y) = E(X^2|Y=y) - (E(X|Y=y))^2$. To compute the previous expected values, one has to derive the conditional density function of X given that $Y=y$. Let's start with the following computations:

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dx = \begin{cases} \int_0^1 \frac{1}{3}(y+x)\,dx, & 0 < y < 2 \\ 0, & \text{otherwise} \end{cases}$$
$$= \begin{cases} \frac{1}{6}(1+2y), & 0 < y < 2 \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} 2 \times \frac{y+x}{1+2y}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

where y is fixed. Now, we can calculate the expected values

$$E(X|Y=y) = \int_{-\infty}^{\infty} x f_{X|Y=y}(x)dx = \int_0^2 2x \times \frac{y+x}{1+2y}dx = \frac{2+3y}{3+6y}$$
$$E(X^2|Y=y) = \int_{-\infty}^{\infty} x^2 f_{X|Y=y}(x)dx = \int_0^2 2x^2 \times \frac{y+x}{1+2y}dx = \frac{3+4y}{6+12y}$$

The usual calculations allow us to get

$$Var(X|Y=y) = E(X^2|Y=y) - (E(X|Y=y))^2 = \frac{1}{36}\left(3 - \frac{1}{(2y+1)^2}\right)$$

7. Let (X, Y) be a discrete random vector with joint probability function given by:

$$f(x, y) = \frac{x + y}{21}, \;\; x = 1, 2; \;\; y = 1, 2, 3$$

(a) Compute the means and variances of X and Y

Solution: $E(X) = 33/21$, $E(Y) = 46/21$, $Var(X) = 12/49$ and $Var(Y) = 278/441$.

(b) Using $E(XY)$ analyze the independence of the two random variables and compute the correlation coefficient.

Solution:$E(XY) = 24/7$. X and Y are not independent. $\rho_{X,Y} = -0.035$

(c) Compute $E(X|Y = 1)$

Solution: $E(X|Y = 1) = 8/5$.

$$P(X = x|Y = 1) = \begin{cases} 2/5, & x = 1 \\ 3/5, & x = 2 \\ 0, & \text{otherwise} \end{cases}$$

8. Suppose that $E(Y|X = x) = 9 + x$. If the $E(X) = 11$ and $Var(X) = 290$, what is $cov(Y, X)$?

Solution: We start by noticing that $Cov(X, Y) = E(XY) - E(X)E(Y)$, which means that we need to discover $E(XY)$ and $E(Y)$. From the tower property, we now that

$$E(Y) = E(E(Y|X)) = E(9 + X) = 9 + E(X) = 20.$$

Moreover,

$$\begin{aligned} E(XY) &= E(E(XY|X)) = E(XE(Y|X)) = E(X(9 + X)) = 9E(X) + E(X^2) \\ &= 99 + (Var(X) + (E(X))^2) = 99 + (290 + 11^2) = 99 + 411 = 510 \end{aligned}$$

Consequently the $Cov(X, Y) = 510 - 11 \times 20 = 290$.

9. Let X and Y be two continuous random variables such that the conditional density function of X given $Y = y$ is

$$f_{X|Y=y}(x) = \begin{cases} \frac{1}{2-y}, & y < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

and the marginal density of Y is given by

$$f_Y(y) = \begin{cases} 1 - y/2, & 0 < y < 2 \\ 0, & \text{otherwise} \end{cases}.$$

Compute the expected value of X.

Solution: Firstly, we can compute

$$E(X|Y=y) = \int_{-\infty}^{+\infty} x f_{X|Y=y}(x)dx = \int_y^2 \frac{x}{2-y}dx = 1 + y/2.$$

Secondly, we can use the tower property

$$E(X) = E(E(X|Y)) = E(1 + Y/2) = 1 + E(Y)/2.$$

By definition,

$$E(Y) = \int_{-\infty}^{+\infty} y f_Y(y)dy = \int_0^2 y(1 - y/2)dy = 2/3.$$

Therefore $E(X) = 1 + 1/3 = 4/3$.

10. Let (X, Y) be a two dimensional random variable, such that its set of discontinuities is $D_{X,Y} = \{(0,1), (0,2), (1,0), (1,1), (1,2)\}$ and its probability function is

$$f_{X,Y}(x,y) = \begin{cases} \frac{x+y}{a}, & (x,y) \in D_{X,Y} \\ 0, & \text{otherwise} \end{cases}.$$

   (a) Find a and represent $f_{X,Y}$ by filling a suitable table.
   Solution: $a = 9$

   (b) Compute $E(X)$ and $E(Y)$.
   Solution: $E(X) = 2/3$ and $E(Y) = 13/9$.

   (c) Calculate $Cov(2X, 3Y)$ and $Var(X + Y)$.
   Solution: $Cov(2X, 3Y) = -4/9$ and $Var(X + Y) = 38/81$

   (d) Characterize the distribution of $E(Y|X)$.
   Solution: If $Z = E[Y|X]$

$$P(Z = z) = \begin{cases} 1/3, & z = 5/3 \\ 2/3, & z = 4/3 \\ 0, & \text{otherwise} \end{cases}$$

11. A company engaged in the trade of various items, whose sales have random behavior. The monthly sales of items A and B, expressed in monetary units, constitute a random vector (X, Y) with joint probability density function given by:

$$f_{X,Y}(x,y) = \frac{1}{2}, \quad 0 < x < 2,\ 0 < y < x.$$

(a) Compute the means and variaces of X and Y.

Solution: For Y we have the following

$$f_Y(y) = \frac{2-y}{2}, \quad 0 < y < 2$$

and

$$E(Y) = \frac{2}{3} \quad \text{and} \quad Var(Y) = 2/9$$

For X we have the following

$$f_X(x) = \frac{x}{2}, \quad 0 < x < 2$$

and

$$E(X) = \frac{4}{3} \quad \text{and} \quad Var(X) = 2/9$$

(b) Analyze the independence of the two random variables and compute the correlation coefficient.

Solution: The random variables are not independent because $f_{X,Y}(x,y) \neq f_X(x) \times f_Y(y)$. Additionally,

$$\rho_{X,Y} = \frac{1 - 2/3 \times 4/3}{2/9} = 1/2.$$

(c) Find the $E(Y|X=1)$.

Solution: For $Y|X=1$ we have the following

$$f_{Y|X=1}(y) = \frac{1/2}{1/2} = 1, \quad 0 < y < 1.$$

Therefore,

$$E(Y|X=1) = \frac{1}{2}$$

(d) Compute the mean and variance of total sales of the two articles.

Solution: Taking into account that

$$E(X+Y) = E(X) + E(Y), \quad Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$$

we get

$$E(X+Y) = 2 \quad \text{and} \quad Var(X+Y) = 2/3$$