ISEG - Lisbon School of Economics and Management

List of Exercises - Chapter 4
1^{st} Semester of 2020/2021

November 5, 2020

1. Let (X, Y) be a two-dimensional random variable such that its cumulative distribution function is given by

$$F_{X,Y}(x,y) = \begin{cases} 0, & x < 0 \text{ or } y < 0 \\ \frac{x+y}{2}, & 0 \le x < 1 \text{ and } 0 \le y < 1 \\ \frac{1+y}{2}, & x \ge 1 \text{ and } 0 \le y < 1 \\ \frac{x+1}{2}, & 0 \le x < 1 \text{ and } y \ge 1 \\ 1, & x \ge 1 \text{ and } y \ge 1 \end{cases}$$

Compute the marginal cumulative distribution functions of X and Y: F_X and F_Y.

Solution: By definition, we know that $F_X(x) = \lim_{y \to +\infty} F_{X,Y}(x,y)$. Therefore, we have to look to the first, fourth and fifth branch of the function $F_{X,Y}$ and take the limit when $y \to +\infty$. As a consequence, we get

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x+1}{2}, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}.$$

Similarly, we can obtain

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{y+1}{2}, & 0 \le y < 1 \\ 1, & y \ge 1 \end{cases}.$$

2. Let X and Y be two independent random variables such that

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{2}, & 0 \le x < 2 \\ 1, & x \ge 2 \end{cases}, \quad \text{and} \quad F_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{y}{3}, & 0 \le y < 3 \\ 1, & y \ge 3 \end{cases}.$$

a) Find the joint cumulative distribution function of X and Y.
Solution: The joint cumulative distribution function is

$$F_{X,Y}(x,y) = F_X(x) \times F_Y(y) = \begin{cases} 0, & x < 0 \text{ or } y < 0 \\ \frac{xy}{6}, & 0 \leq x < 2, 0 \leq y < 3 \\ \frac{x}{2}, & 0 \leq x < 2, y \geq 3 \\ \frac{y}{3}, & 0 \leq y < 3, x \geq 2 \\ 1, & x \geq 2, y \geq 3 \end{cases}.$$

b) Compute the marginal density function of X and Y
Solution: To compute the density function of X and Y, one has to calculate the derivative of F_X and F_Y. Therefore,

$$f_X(x) = \begin{cases} \frac{1}{2}, & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} \frac{1}{3}, & 0 < y < 3 \\ 0, & \text{otherwise} \end{cases}.$$

3. If the values of the joint probability function of X and Y are as shown in the table

Y \ X	0	1	2
0	1/12	1/6	1/24
1	1/4	1/4	1/40
2	1/8	1/20	0
3	1/120	0	0

(a) find:

i. $P(X = 1, Y = 2)$;
Answer: 1/20

ii. $P(X = 0, 1 \leq Y < 3)$;
Answer: 3/8

iii. $P(X + Y \leq 1)$;
Answer: 1/2

iv. $P(X > Y)$.
Answer: 7/30

(b) find the following values of the joint cumulative distribution function of the two random variables:

i. $F(1.2, 0.9)$;
Answer: 1/4

ii. $F(-3, 1.5)$;
Answer: 0

iii. $F(2, 0)$;
Answer: 7/24

iv. $F(4, 2.7)$.
Answer: 119/120

4. If the joint probability distribution of X and Y is given by

$$f_{X,Y}(x, y) = c(x^2 + y^2), \quad \text{for } x = 1, 3; \quad y = -1, 2.$$

a) Find the value of c;
Answer: Since $f_{X,Y}$ is a joint probability distribution we have that $f_{X,Y}(x, y) \geq 0$ for all $(x, y) \in \mathbb{R}^2$ and $\sum_{(x,y)\in D_{X,Y}} f_{X,Y}(x, y) = 1$. From the first condition one gets that $c \geq 0$ and, from the second one, we have

$$f_{X,Y}(1, -1) + f_{X,Y}(1, 2) + f_{X,Y}(3, -1) + f_{X,Y}(3, 2) = 1 \Leftrightarrow c = \frac{1}{30}.$$

b) Compute $P(X + Y > 2)$;
Answer:

$$P(X + Y > 2) = P(X = 1, Y = 2) + P(X = 3, Y = 2) = \frac{5}{30} + \frac{13}{30} = \frac{3}{5}.$$

c) Compute the cumulative distribution function.

Answer: The cumulative distribution function is given by

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y) = \begin{cases} 0, & x < 1 \text{ or } y < -1 \\ \frac{1}{15}, & 1 \leq x < 3, \ -1 \leq y < 2 \\ \frac{7}{30}, & 1 \leq x < 3, \ y \geq 2 \\ \frac{2}{5}, & x \geq 3, \ -1 \leq y < 2 \\ 1, & x \geq 3, \ y \geq 2 \end{cases}.$$

5. Given the values of the joint probability distribution of X and Y shown in the table

	$X = -1$	$X = 1$
$Y = -1$	$\frac{1}{8}$	$\frac{1}{2}$
$Y = 0$	0	$\frac{1}{4}$
$Y = 1$	$\frac{1}{8}$	0

a) Find the marginal probability function of X.

Answer: The marginal probability function of X is given by

$$f_X(x) = \sum_{y=-1}^{1} P(X = x, Y = y) = \begin{cases} \frac{2}{8}, & x = -1 \\ \frac{3}{4}, & x = 1 \\ 0, & \text{elsewhere} \end{cases}.$$

b) Find the marginal probability function of Y.

Answer: The marginal probability function of Y is given by

$$f_Y(y) = P(X = -1, Y = y) + P(X = 1, Y = y) = \begin{cases} \frac{5}{8}, & y = -1 \\ \frac{1}{4}, & y = 0 \\ \frac{1}{8}, & y = 1 \\ 0, & \text{elsewhere} \end{cases}.$$

c) Find the conditional probability function of X given $Y = -1$.

Answer: The conditional probability function of X given $Y = -1$ is

$$f_{X|Y=-1}(x) = \frac{P(X = x, Y = -1)}{P(Y = -1)} = \begin{cases} \frac{1}{5}, & x = -1 \\ \frac{4}{5}, & x = 1 \\ 0, & \text{elsewhere} \end{cases}.$$

d) Compute the conditional cumulative distribution function of X given $Y = -1$.

Answer: The conditional cumulative distribution function of X given $Y = -1$ is

$$F_{X|Y=-1}(x) = P(X \leq x | Y = -1) = \sum_{x' \leq x} f_{X|Y=-1}(x') = \begin{cases} 0, & x < -1 \\ \frac{1}{5}, & -1 \leq x < 1 \\ 1, & x \geq 1 \end{cases}.$$

e) Verify if X and Y are independent.

Answer: X and Y are not independent random variables since $f_X(x) \neq f_{X|Y=-1}(x)$.

6. If the values of the joint probability function of X and Y are as shown in the table

Y \ X	0	1	2
0	1/12	1/6	1/24
1	1/4	1/4	1/40
2	1/8	1/20	0
3	1/120	0	0

find

a) the marginal probability function of X;

Answer:

$$P(X=x) = \begin{cases} 7/15, & x = 0, 1 \\ 1/15, & x = 2 \\ 0, & otherwise \end{cases}$$

b) the marginal probability function of Y;

Answer:

$$P(Y=y) = \begin{cases} 7/24, & y = 0 \\ 21/40, & y = 1 \\ 7/40, & y = 2 \\ 1/120, & y = 3 \\ 0, & otherwise \end{cases}$$

c) the conditional probability function of X given $Y = 1$;

Answer:

$$P(X=x|Y=1) = \begin{cases} 10/21, & x = 0, 1 \\ 1/21, & x = 2 \\ 0, & otherwise \end{cases}$$

d) the conditional probability function of Y given $X = 0$.

Answer:

$$P(Y=y|X=0) = \begin{cases} 5/28, & y = 0 \\ 15/28, & y = 1 \\ 15/56, & y = 2 \\ 1/56, & y = 3 \\ 0, & otherwise \end{cases}$$

7. If the joint cumulative distribution function of X and Y is given by

$$F_{X,Y}(x,y) = \begin{cases} (1-e^{-x^2})(1-e^{-y^2}) & \text{, for } x \geq 0,\ y \geq 0 \\ 0 & \text{, elsewhere} \end{cases}$$

(a) Find the marginal cumulative distribution functions of the two random variables X and Y.

Answer:

$$F_X(x) = \lim_{y \to +\infty} F_{X,Y}(x,y) = \begin{cases} (1-e^{-x^2}), & \text{for } x \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

and

$$F_Y(y) = \lim_{x \to +\infty} F_{X,Y}(x,y) = \begin{cases} (1-e^{-y^2}), & \text{for } y \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

(b) Find the joint density function of the two random variables X and Y.

Answer:

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y) = \begin{cases} 4xye^{-(x^2+y^2)} & \text{, for } x > 0,\ y > 0 \\ 0 & \text{, elsewhere} \end{cases}$$

(c) Find $P(1 < X \leq 2, 1 < Y \leq 2)$.

Answer:

$$\begin{aligned} P(1 < X \leq 2, 1 < Y \leq 2) &= F_{X,Y}(2,2) - F_{X,Y}(1,2) - F_{X,Y}(2,1) + F_{X,Y}(1,1) \\ &= \frac{(e^3-1)^2}{e^8} \end{aligned}$$

(d) Verify if X and Y are independent random variables.

Answer: The random variables X and Y are independent because $F_{X,Y}(x,y) = F_X(x) \times F_Y(y)$.

8. If the joint probability density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4}(2x+y) & \text{, for } 0 < x < 1,\ 0 < y < 2 \\ 0 & \text{, elsewhere} \end{cases}$$

find

a) the marginal density of X;

Answer: The marginal density of X is given by

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy = \begin{cases} \frac{1}{4}(4x+2), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

b) the conditional density of Y given $X = 1/4$

Answer: The conditional density of Y given $X = 1/4$ is given by

$$f_{Y|X=1/4}(y) = \begin{cases} \frac{f_{X,Y}(1/4,y)}{f_X(1/4)}, & 0 < y < 2 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{3}(y+\frac{1}{2}), & 0 < y < 2 \\ 0, & \text{otherwise} \end{cases}$$

c) the marginal density of Y;

Answer: The marginal density of Y is given by

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dx = \begin{cases} \frac{1}{4}(1+y), & 0 < y < 2 \\ 0, & \text{otherwise} \end{cases}$$

d) the conditional density of X given $Y = 1$;

Answer: The conditional density of X given $Y = 1$ is given by

$$f_{X|Y=1}(x) = \begin{cases} \frac{f_{X,Y}(x,1)}{f_Y(1)}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{2}(2x+1), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

e) $P(X > \frac{1}{4}|X < \frac{3}{4})$.

Answer:

$$P\left(X > \frac{1}{4}\Big|X < \frac{3}{4}\right) = \frac{P(1/4 < X < 3/4)}{P(X < 3/4)} = \frac{\int_{1/4}^{3/4} f_X(x)dx}{\int_{-\infty}^{3/4} f_X(x)dx} = \frac{1/2}{21/32} = \frac{16}{21}$$

f) the joint cumulative distribution function of X and Y.

Answer: The joint CDF is given by

$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,t)dtdu = \begin{cases} 0, & x < 0 \text{ or } y < 0 \\ \frac{1}{4}(yx^2 + xy^2/2), & 0 \le x < 1,\ 0 \le y < 2 \\ \frac{1}{4}(y + y^2/2), & x \ge 1,\ 0 \le y < 2 \\ \frac{1}{2}(x^2 + x), & 0 \le x < 1,\ y \ge 2 \\ 1, & x \ge 1,\ y \ge 2 \end{cases}$$

9. Consider the joint probability density function of X and Y given by

$$f_{X,Y}(x,y) = \begin{cases} kxy & \text{, for } 0 < x < 2,\ 0 < y < 1 \\ 0 & \text{, elsewhere} \end{cases}$$

a) Determine k so that $f_{X,Y}$ can serve as a joint probability density function.

Answer: If $f_{X,Y}$ is a joint probability density function, then $f_{X,Y}(x,y) \ge 0$ for all $(x,y) \in \mathbb{R}^2$ and $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)dxdy = 1$. From the first condition we get that $k \ge 0$ and from the second one we have that

$$\int_0^2\int_0^1 f_{X,Y}(x,y)dydx = 1 \Leftrightarrow k = 1.$$

b) Determine the joint distribution function of X and Y.

Answer: The joint CDF is given by

$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,t)dtdu = \begin{cases} 0, & x < 0 \text{ or } y < 0 \\ \frac{x^2y^2}{4}, & 0 \le x < 2,\ 0 \le y < 1 \\ y^2, & x \ge 2,\ 0 \le y < 1 \\ \frac{x^2}{4}, & 0 \le x < 2,\ y \ge 1 \\ 1, & x \ge 2,\ y \ge 1 \end{cases}$$

c) Determine the marginal cumulative distribution function of X and Y.

Answer: The marginal CDF of X is given by

$$F_X(x) = \lim_{y\to\infty} F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{+\infty} f_{X,Y}(u,y)dydu = \begin{cases} 0, & x < 0 \\ \frac{x^2}{4}, & 0 \le x < 2 \\ 1, & x \ge 2, \end{cases}$$

The marginal CDF of Y is given by

$$F_Y(y) = \lim_{x\to\infty} F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{+\infty} f_{X,Y}(x,t)dxdt = \begin{cases} 0, & y < 0 \\ y^2, & 0 \le y < 1 \\ 1, & y \ge 1, \end{cases}$$

d) Determine the marginal probability density function of X and Y.

Answer: The marginal probability density function of X is given by

$$f_X(x) = F_X'(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy = \begin{cases} \frac{x}{2}, & 0 < x < 2 \\ 0, & \text{elsewhere}, \end{cases}$$

The marginal probability density function of Y is given by

$$f_Y(y) = F_Y'(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dx = \begin{cases} 2y, & 0 < y < 1 \\ 0, & \text{elsewhere}, \end{cases}$$

e) Verify if X and Y are independent.

Answer: The random variables are independent because $F_{X,Y}(x,y) = F_X(x)F_Y(y)$.

f) Compute $P(X < 1.25, Y \le 0.5)$, $P\left(Y > \frac{X}{4} + \frac{1}{2}\right)$ and $P\left(Y < \frac{X}{4} + \frac{1}{2}\right)$.

Answer:

$$P(X < 1.25, Y \le 0.5) = F_{X,Y}(1.25, 0.5) = \frac{25}{256}$$

$$P\left(Y > \frac{X}{4} + \frac{1}{2}\right) = \int_0^2 \int_{\frac{x}{4}+\frac{1}{2}}^{1} f_{X,Y}(x,y)dydx = \frac{7}{24}$$

$$P\left(Y < \frac{X}{4} + \frac{1}{2}\right) = 1 - P\left(Y \ge \frac{X}{4} + \frac{1}{2}\right) = 1 - P\left(Y > \frac{X}{4} + \frac{1}{2}\right) = \frac{17}{24}$$

g) Compute $f_{X|Y=y}$ and $f_{Y|X=x}$.

Answer: The conditional probability density function $f_{X|Y=y}$ and $f_{Y|X=x}$ are given by

$$f_{X|Y=y}(x) = \begin{cases} \frac{f_{X,Y}(x,y)}{f_Y(y)}, & 0 < x < 2 \\ 0, & \text{elsewhere}, \end{cases} = \begin{cases} \frac{x}{2}, & 0 < x < 2 \\ 0, & \text{elsewhere}, \end{cases}$$

and

$$f_{Y|X=x}(y) = \begin{cases} \frac{f_{X,Y}(x,y)}{f_X(x)}, & 0 < y < 1 \\ 0, & \text{elsewhere}, \end{cases} = \begin{cases} 2y, & 0 < y < 1 \\ 0, & \text{elsewhere}, \end{cases}$$

10. Let X and Y be two independent random variables such that

$$f_X(x) = \begin{cases} \frac{1}{2}, & 0 < x < 2 \\ 0, & \text{elsewhere} \end{cases}, \quad \text{and} \quad f_Y(y) = \begin{cases} \frac{1}{3}, & 0 < y < 3 \\ 0, & \text{elsewhere} \end{cases}.$$

Find the joint density function of X and Y.

Answer:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{6}, & 0 < x < 2, 0 < y < 3 \\ 0, & \text{elsewhere} \end{cases}$$

11. If the joint probability density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 24xy & \text{, for } 0 < x < 1,\ 0 < y < 1,\ x+y < 1 \\ 0 & \text{, elsewhere} \end{cases}$$

Compute $P(X + Y < 1/2)$.

Answer:

$$P(X+Y < \frac{1}{2}) = \int_0^{1/2} \int_0^{1/2-x} 24xy\,dy\,dx = 12 \int_0^{1/2} (1/2 - x)^2 x\,dx = \frac{1}{16}$$

12. If the joint probability density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 24y\,(1 - x - y) & \text{, for } x > 0, y > 0, x+y < 1 \\ 0 & \text{, elsewhere} \end{cases}$$

a) Find the marginal density of X;

Answer:

$$f_X(x) = \begin{cases} 4(1-x)^3 & \text{, for } 0 < x < 1 \\ 0 & \text{, elsewhere} \end{cases}$$

b) Find the marginal density of Y;

Answer:

$$f_Y(y) = \begin{cases} 12y(1-y)^2 & \text{, for } 0 < y < 1 \\ 0 & \text{, elsewhere} \end{cases}$$

c) Determine whether the two random variables are independent.

Answer: X and Y are not independent random variables.