ISEG - Lisbon School of Economics and Management

List of Exercises - Chapter 4
1^{st} Semester of 2020/2021

March 25, 2021

1. Let (X, Y) be a two-dimensional random variable such that its cumulative distribution function is given by

$$F_{X,Y}(x,y) = \begin{cases} 0, & x < 0 \text{ or } y < 0 \\ \frac{x+y}{2}, & 0 \leq x < 1 \text{ and } 0 \leq y < 1 \\ \frac{1+y}{2}, & x \geq 1 \text{ and } 0 \leq y < 1 \\ \frac{x+1}{2}, & 0 \leq x < 1 \text{ and } y \geq 1 \\ 1, & x \geq 1 \text{ and } y \geq 1 \end{cases}$$

Compute the marginal cumulative distribution functions of X and Y: F_X and F_Y.

2. Let X and Y be two independent random variables such that

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{2}, & 0 \leq x < 2 \\ 1, & x \geq 2 \end{cases}, \quad \text{and} \quad F_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{y}{3}, & 0 < y < 3 \\ 1, & y \geq 3 \end{cases}.$$

   a) Find the joint density function of X and Y.
   b) Compute the marginal density function of X and Y

3. If the values of the joint probability function of X and Y are as shown in the table

Y \ X	0	1	2
0	1/12	1/6	1/24
1	1/4	1/4	1/40
2	1/8	1/20	0
3	1/120	0	0

(a) find:

i. $P(X = 1, Y = 2)$;

ii. $P(X = 0, 1 \leq Y < 3)$;

iii. $P(X + Y \leq 1)$;

iv. $P(X > Y)$.

(b) find the following values of the joint cumulative distribution function of the two random variables:

i. $F(1.2, 0.9)$;

ii. $F(-3, 1.5)$;

iii. $F(2, 0)$;

iv. $F(4, 2.7)$.

4. If the joint probability distribution of X and Y is given by

$$f_{X,Y}(x, y) = c(x^2 + y^2), \quad \text{for } x = 1, 3; \quad y = -1, 2.$$

a) Find the value of c;

b) Compute $P(X + Y > 2)$;

c) Compute the cumulative distribution function.

5. Given the values of the joint probability distribution of X and Y shown in the table

	$X = -1$	$X = 1$
$Y = -1$	$\frac{1}{8}$	$\frac{1}{2}$
$Y = 0$	0	$\frac{1}{4}$
$Y = 1$	$\frac{1}{8}$	0

a) Find the marginal probability function of X.

b) Find the marginal probability function of Y.

c) Find the conditional probability function of X given $Y = -1$.

d) Compute the conditional cumulative distribution function of X given $Y = -1$.

e) Verify if X and Y are independent.

6. If the values of the joint probability function of X and Y are as shown in the table

Y \ X	0	1	2
0	1/12	1/6	1/24
1	1/4	1/4	1/40
2	1/8	1/20	0
3	1/120	0	0

find

a) the marginal probability function of X;
b) the marginal probability function of Y;
c) the conditional probability function of X given $Y = 1$;
d) the conditional probability function of Y given $X = 0$.

7. If the joint cumulative distribution function of X and Y is given by

$$F_{X,Y}(x,y) = \begin{cases} (1-e^{-x^2})(1-e^{-y^2}) & \text{, for } x \geq 0,\ y \geq 0 \\ 0 & \text{, elsewhere} \end{cases}$$

(a) Find the marginal cumulative distribution functions of the two random variables X and Y.
(b) Find the joint density function of the two random variables X and Y.
(c) Find $P(1 < X \leq 2, 1 < Y \leq 2)$.
(d) Verify if X and Y are independent random variables.

8. If the joint probability density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4}(2x+y) & \text{, for } 0 < x < 1,\ 0 < y < 2 \\ 0 & \text{, elsewhere} \end{cases}$$

find

a) the marginal density of X;
b) the conditional density of Y given $X = 1/4$
c) the marginal density of Y;
d) the conditional density of X given $Y = 1$;
e) $P(X > \frac{1}{4} | X < \frac{3}{4})$.
f) the joint cumulative distribution function of X and Y.

9. Consider the joint probability density function of X and Y given by

$$f_{X,Y}(x,y) = \begin{cases} kxy & \text{, for } 0 < x < 2,\ 0 < y < 1 \\ 0 & \text{, elsewhere} \end{cases}$$

a) Determine k so that $f_{X,Y}$ can serve as a joint probability density function.
b) Determine the joint distribution function of X and Y.
c) Determine the marginal cumulative distribution function of X and Y.

d) Determine the marginal probability density function of X and Y.

e) Verify if X and Y are independent.

f) Compute Compute $P(X < 1.25, Y \leq 0.5)$, $P\left(Y > \frac{X}{4} + \frac{1}{2}\right)$ and $P\left(Y < \frac{X}{4} + \frac{1}{2}\right)$.

g) Compute $f_{X|Y=y}$ and $f_{Y|X=x}$.

10. Let X and Y be two independent random variables such that

$$f_X(x) = \begin{cases} \frac{1}{2}, & 0 < x < 2 \\ 0, & \text{elsewhere} \end{cases}, \quad \text{and} \quad f_Y(y) = \begin{cases} \frac{1}{3}, & 0 < y < 3 \\ 0, & \text{elsewhere} \end{cases}.$$

Find the joint density function of X and Y.

11. If the joint probability density of X and Y is given by

$$f_{X,Y}(x, y) = \begin{cases} 24xy & \text{, for } 0 < x < 1,\ 0 < y < 1,\ x + y < 1 \\ 0 & \text{, elsewhere} \end{cases}$$

a) Compute $P(X + Y < 1/2)$.

b) Compute $F_{X|Y=\frac{1}{2}}(x)$

12. If the joint probability density of X and Y is given by

$$f_{X,Y}(x, y) = \begin{cases} 24y\,(1 - x - y) & \text{, for } x > 0, y > 0, x + y < 1 \\ 0 & \text{, elsewhere} \end{cases}$$

a) Find the marginal density of X;

b) Find the marginal density of Y;

d) Determine whether the two random variables are independent.

13. If X is the amount of money (in dollars) that a salesperson spends on gasoline during a day and Y is the corresponding amount of money (in dollars) for which he or she is reimbursed, the joint density of these two random variables is given by

$$f(x, y) = \begin{cases} \frac{1}{25}\left(\frac{20-x}{x}\right) & \text{, for } 10 < x < 20, \frac{x}{2} < y < x \\ 0 & \text{, elsewhere} \end{cases}$$

find

(a) the marginal density of X;

(b) the conditional density of Y given $X = 12$;

(c) the probability that the salesperson will be reimbursed at least \$8 when spending \$12.