ISEG - Lisbon School of Economics and Management

List of Exercises - Chapter 3
1st Semester of 2020/2021

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1. Let X be a random variable that takes on the values 0, 1, 2, and 3 with probabilities $\frac{1}{10}$, $\frac{3}{10}$, $\frac{2}{10}$, $\frac{4}{10}$.

   (a) Find $E(X)$ and $E(X^2)$.

   Answer: By definition,

$$E(X) = \sum_{i=0}^{3} i f_X(i) = 0 \times \frac{1}{10} + 1 \times \frac{3}{10} + 2 \times \frac{2}{10} + 3 \times \frac{4}{10} = \frac{19}{10}.$$

   and

$$E(X^2) = \sum_{i=0}^{3} i^2 f_X(i) = 0^2 \times \frac{1}{10} + 1^2 \times \frac{3}{10} + 2^2 \times \frac{2}{10} + 3^2 \times \frac{4}{10} = \frac{47}{10}.$$

   (b) Use the results of part *(a)* to determine the value of $E[(X-\mu_X)^2]$.

   Answer: One can easily notice that $E[(X-\mu_X)^2] = \sigma_X^2 = E(X^2) - (E(X))^2 = \frac{109}{100}$.

   (c) Use the definition to calculate σ_X.

   Answer: One can easily see that:

$$\sigma_X^2 = \sum_{i=0}^{3} (i - \mu_X) f_X(i) = \frac{109}{100}.$$

   Then,

$$\sigma_X = \sqrt{\sigma_X^2} = \sqrt{\frac{109}{100}}.$$

2. Let X be a continuous random variable and f_X its density function

$$f_X(x) = \begin{cases} 1/3, & 0 < x < 1 \\ 4/45x, & 1 < x < 4 \end{cases}.$$

   (a) Compute the expected value and the variance of X.
   Solution: $E(X) = 61/30$ and $Var(X) = 493/300$.

   (b) Compute the expected value of Y that is given by

$$Y = g(X) = \begin{cases} 0, & X < 1 \\ 1, & X \geq 1 \end{cases}.$$

   Solution: By definition, the expected value of $g(X)$ is given by

$$E(X) = \int_0^4 g(x) \times f(x)dx = 0 \times \int_0^1 \times 1/3dx + 1 \times \int_1^4 \frac{4}{45}xdx = 2/3.$$

   (c) Compute the expected value of $Z = 2Y - 1$.
   Solution: By using the properties of the expected value, we get

$$E(Z) = 2 \times E(Y) - 1 = 1/3.$$

3. The demand of a certain product, in Kg, in a random day is well represented by the random variable X with density function

$$f_X(x) = \begin{cases} 1/5, & 0 < x < 5 \\ 0, & \text{otherwise} \end{cases}.$$

The firm that sells this product has a profit of 5 euros per Kg sold and a loss of 2 euros per Kg that is not sold.

   (a) How many Kg of the product should the firm have in stock to maximize the expected profit?
   Solution: Let y be the stock that we intend maximize. Then, the profit of the firm is given my

$$\begin{aligned} P &= 5 \times \min(y, X) - 2 \times \max(0, y - X) \\ &= \begin{cases} 5X - 2(y - X), & X < y \\ 5y, & X \geq y \end{cases} \end{aligned}$$

Then the expected profit is $E(P) = 5E(\min(y,X)) - 2E(\max(0, y - X))$. Since

$$E(\min(y,X)) = \int_{-\infty}^{\infty} \min(y,x) f_X(x)dx = \frac{1}{5}\int_0^y xdx + \frac{1}{5}\int_y^5 ydx = \frac{y^2/2 + y(5-y)}{5}$$

$$E(\max(0,y-X)) = \int_{-\infty}^{\infty} \max(0,y-x) f_X(x)dx = \frac{1}{5}\int_0^y y - xdx = \frac{y^2 - y^2/2}{5}.$$

Therefore,

$$E(L) = 5y - 7y^2/10.$$

Therefore, the expected profit follows from

$$\frac{\partial E(P)}{\partial y} = 0 \Leftrightarrow 5 - 7/5y = 0 \Leftrightarrow y = 25/7$$

because

$$\frac{\partial^2 E(P)}{\partial y^2} = -7/5 < 0.$$

Equivalently, we could see that

$$L = \begin{cases} 7X - 2y, & X < y \\ 5y, & X \geq y \end{cases}.$$

(b) Assume now that X is a discrete random variable, with a probability function

$$f_X(x) = \frac{1}{6}, \quad \text{for } x = 0, 1, 2, 3, 4, 5$$

and solve question (a).

Solution: It is known that

$$L = \pi(X,y) = \begin{cases} 7X - 2y, & X \leq y \\ 5y, & X \geq y \end{cases}.$$

Then the expected value of L is given by

$$E(L) = \sum_{x=0}^{5} \pi(x,y) f_X(x) = \frac{1}{6}\sum_{x=0}^{5} \pi(x,y) = \frac{1}{6}\left(\sum_{x=0}^{y}(7x - y) + \sum_{x=y+1}^{5} 5y\right)$$

$$= \frac{1}{6}\left(7 \times \frac{y(y+1)}{2} - 2y(y+1) + 5y(5-y)\right) = \frac{1}{6}\left(-7/2y^2 + 53/2y\right).$$

It is not possible to compute derivatives, therefore we can notice that

$$E(\pi(X,0)) = 0, \quad E(\pi(X,1)) = \frac{23}{6}, \quad E(\pi(X,2)) = \frac{13}{2}, \quad E(\pi(X,3)) = 8$$

$$E(\pi(X,4)) = \frac{25}{3} \quad \textit{and} \quad E(\pi(X,5)) = \frac{15}{2}.$$

4. Let X be a continuous random variable and f_X its distribution. Assume that a and b are constants and prove that the expected value of

$$Y = \begin{cases} a, & X < 0 \\ b, & X \geq 0 \end{cases},$$

is $E(Y) = aP(X < 0) + bP(X \geq 0)$.

Solution: By definition of expected value, we have that

$$\begin{aligned} E(Y) &= \int_{-\infty}^{0} a f_X(x)dx + \int_{0}^{\infty} b f_X(x)dx \\ &= a \int_{-\infty}^{0} f_X(x)dx + b \int_{0}^{\infty} f_X(x)dx \\ &= aP(X < 0) + bP(X \geq 0). \end{aligned}$$

5. Find $E(X)$, $E(X^2)$ and σ_X^2 for the random variable X that has probability density function

$$f_X(x) = \begin{cases} \frac{x}{2} & \text{for } 0 < x < 2 \\ 0 & \text{elsewhere} \end{cases}.$$

Answer: To solve this exercise, one has to use the definitions of $E(X)$, $E(X^2)$ and σ_X^2. Thus,

$$\begin{aligned} E(X) &= \int_{-\infty}^{+\infty} x f_X(x)dx = \int_{0}^{2} \frac{x^2}{2}dx = \frac{4}{3}. \\ E(X^2) &= \int_{-\infty}^{+\infty} x^2 f_X(x)dx = \int_{0}^{2} \frac{x^3}{2}dx = 2. \\ \sigma_X^2 &= E(X^2) - (E(X))^2 = \frac{2}{9}. \end{aligned}$$

6. Let X be a discrete random variable such that

$$f_X(x) = \begin{cases} 1/2, & x = 0 \\ 1/3, & x = 1 \\ 1/6, & x = 2 \end{cases}$$

Compute γ_1.

Answer: We know that $\gamma_1 = \frac{E\left[(X-\mu_X)^3\right]}{Var(X)^{3/2}} = \frac{\mu_3}{\sigma_X^3}$. Then, we may start by computing

$$\mu_X = E(X) = \sum_{x=0}^{2} x \times f_X(x) = \frac{2}{3}.$$

Additionally,

$$E\left[(X-\mu_X)^3\right] = \sum_{x=0}^{2}\left(x-\frac{2}{3}\right)^3 \times f_X(x) = \frac{7}{27}$$

Taking into account that

$$E(X^2) = \sum_{x=0}^{2} x^2 \times f_X(x) = 1,$$

Then, $Var(X) = E(X^2) - (E(X))^2 = 1 - \frac{4}{9} = \frac{5}{9}$. As a conclusion, $\gamma_1 \approx 0.383633$.

7. Let X be a continuous random variable such that

$$f_X(x) = \begin{cases} x, & 0 < x < 1 \\ 1/2, & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Compute γ_2.

Answer: We know that $\gamma_2 = \frac{E\left[(X-\mu_X)^4\right]}{Var(X)^2} = \frac{\mu_4}{\sigma_X^4}$. Since

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f_X(x)dx = \int_0^1 x^2 dx + \int_1^2 \frac{x}{2}dx = \frac{13}{12}, \\ E(X^2) &= \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_0^1 x^3 dx + \int_1^2 \frac{x^2}{2}dx = \frac{17}{12}, \\ Var(X) &= E(X^2) - (E(X))^2 = 1/3 \\ E\left[(X-\mu_X)^4\right] &= \int_{-\infty}^{\infty} (x-\mu_X)^4 f_X(x)dx = \int_0^1 x(x-\mu_X)^4 dx + \int_1^2 \frac{1}{2}(x-\mu_X)^4 dx \\ &= 0.1186, \end{aligned}$$

one may notice that $\gamma_2 = \frac{0.1186^2}{1/9} \approx 1.067$.

8. Let X be a random variable that has probability density function

$$f(x) = \begin{cases} x/2 & \text{for } 0 < x \le 1 \\ 1/2 & \text{for } 1 < x \le 2 \\ (3-x)/2 & \text{for } 2 < x < 3 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find $E(X)$, the median and the mode of X.

Answer: $E(X) = \frac{3}{2}$.

(b) Find $E\left(X^2\right)$.

Answer: $E(X^2) = \frac{8}{3}$

(c) Use the results of part (a) and (b) to determine $E\left(X^2 - 5X + 3\right)$.

Answer: $E\left(X^2 - 5X + 3\right) = -\frac{11}{6}$

(d) Compute the standard deviation.

Answer: $\sigma_X = \sqrt{\frac{5}{12}}$.

9. Find the expected value, the median and the mode of the discrete random variable X having the probability distribution $f_X(x) = |x-2|/7$, $x = -1, 0, 1, 3$.

Answer: Expected value:

$$E(X) = \sum_{j=1}^{4} x_j f_X(x_j) = -1 \times \frac{3}{7} + 0 \times \frac{2}{7} + 1 \times \frac{1}{7} + 3 \times \frac{3}{7} = \frac{1}{7}.$$

Mode:

$$mo(X) = \arg\max_{x\in\mathbb{R}} f_X(x) = \arg\max_{x\in\mathbb{R}} P(X = x) = -1.$$

Median:

$$me(X) = \min\{x \in \mathbb{R} : F_X(x) \geq 0.5\} = 0$$

because,

$$F_X(x) = \begin{cases} 0, & x < -1 \\ 3/7, & -1 \leq x < 0 \\ 5/7, & 0 \leq x < 1 \\ 6/7, & 1 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

10. Find the expected value, the median and the mode, of the random variable Y whose probability density is given by

$$f_Y\left(y\right) = \begin{cases} \left(y+1\right)/8 & \text{for } 2 \leq y \leq 4 \\ 0 & \text{elsewhere} \end{cases}$$

Answer: Expected value:

$$E(Y) = \int_{-\infty}^{+\infty} y f_Y(y) dy = \int_2^4 \frac{y}{8}(y+1) dy = \frac{37}{12}$$

Mode:

$$mo(Y) = \arg\max_{y\in\mathbb{R}} f_Y(y) = 4,$$

because the density function is strictly increasing and positive in the interval $(2,4)$.

Median:

$$me(X) : F_X(me(X)) = 0.5 \Leftrightarrow me(X) = \sqrt{17} - 1$$

because,

$$F_X(x) = \begin{cases} 0, & x < 2 \\ y^2/16 + y/8 - 1/2, & 2 \leq x < 4 \\ 1, & x \geq 4 \end{cases}$$

11. Let X be a random variable that has the probability function $f_X(x) = 1/2$ for for $x = -2$ and $x = 2$.

   (a) Find $E(X)$, $E(X^2)$ and σ_X^2.

   Answer:

$$E(X) = -2f_X(-2) + 2f_X(2) = 0$$
$$E(X^2) = (-2)^4 f_X(-2) + 2^2 f_X(2) = 4$$
$$\sigma_X^2 = E(X^2) - (E(X))^2 = 4.$$

   (b) Calculate the the mode and median.

   Answer: Mode:

$$mo(X) = \arg\max_{x \in \mathbb{R}} f_X(x) = \arg\max_{x \in \mathbb{R}} P(X = x) = -2 \text{ and } 2$$

   Median:

$$me(X) = \min\{x \in \mathbb{R} : F_X(x) \geq 0.5\} = -2,$$

   because

$$F_X(x) = \begin{cases} 0, & x < -2 \\ \frac{1}{2}, & -2 \leq x < 2 \\ 1, & x \geq 2 \end{cases}.$$

   (c) Calculate first and third quartiles.

   First quartile:

$$q_{0.25}(X) = \min\{x \in \mathbb{R} : F_X(x) \geq 0.25\} = -2.$$

   Third quartile:

$$q_{0.75}(X) = \min\{x \in \mathbb{R} : F_X(x) \geq 0.75\} = 2.$$

   (d) Compute the standard deviation.

   Answer: $\sigma_X = 2$.

(e) Compute $Var(2X-2)$

Answer: 16.

12. Let X be a random variable that has probability density function

$$f_X(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2-x & \text{for } 1 \leq x < 2 \\ 0 & \text{elsewhere} \end{cases}.$$

(a) Find the expected value, the median and the mode of the random variable X.

Answer: Expected value:

$$E(X) = \int_{-\infty}^{+\infty} x f_X(x)dx = \int_0^1 x^2 dx + \int_1^2 2x - x^2 dx = 1$$

Mode:

$$mo(X) = \arg\max_{x\in\mathbb{R}} f_X(x) = 1,$$

because f_X is an increasing function in $(0,1)$ and a decreasing function in $[1,2)$. Outside of $(0,2)$ the function is constant and equals zero.

Median:

$$me(X) : F_X(me(X)) = 0.5 \Leftrightarrow me(X) = 1,$$

because

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{2}, & 0 \leq x < 1 \\ 2x - 1 - \frac{x^2}{2}, & 1 \leq x < 2 \\ 1, & x \geq 2 \end{cases}.$$

(b) Compute the variance of $g(X) = 2X + 3$.

Answer: Taking into account the properties of the variance, we have that $Var(g(X)) = 4Var(X)$. Then,

$$Var(2X+3) = 4Var(X) = 4(E(X^2) - E^2(X)).$$

Since

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f_X(x)dx = \int_0^1 x^3 dx + \int_1^2 2x^2 - x^3 dx = \frac{7}{6},$$

then,

$$Var(2X+3) = 4Var(X) = 4(E(X^2) - E^2(X)) = 4(7/6 - 1) = 2/3.$$

13. Let X be a random variable that has probability density function

$$f(x) = \begin{cases} \frac{1}{x\log(3)} & \text{for } 1 \leq x \leq 3 \\ 0 & \text{elsewhere} \end{cases}.$$

(a) Find $E(X)$, the median and the mode of X.
Answer: $E(X) = \frac{2}{\log(3)}$, $mo(X) = 1$ and $me(X) = \sqrt{3}$.

(b) Find $E(X^2)$ and $E(X^3)$.
Answer: $E(X^2) = \frac{4}{\log(3)}$. and $E(X^3) = \frac{26}{3\log(3)}$.

(c) Use the results of part (a) and (b) to determine $E(X^3 + 2X^2 - 3X + 1)$
Answer: $E(X^3 + 2X^2 - 3X + 1) = 1 + \frac{32}{3\log(3)}$

(d) Compute the standard deviation.
Answer: $\sigma_X = \frac{2\sqrt{\log(3)-1}}{(\log(3))^2}$.

14. Let X be a random variable such that

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0 \end{cases}$$

(a) Find the moment generating function of X.
Answer: The moment generating function is, by definition given by

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{+\infty} e^{tx} f_X(x)dx = \frac{1}{b-a}\int_a^b e^{tx}dx$$
$$= \begin{cases} \frac{1}{t(b-a)}(e^{bt} - e^{at}), & t > 0 \\ 1, & t = 0 \end{cases}$$

(b) Calculate the first and third quantiles.
Answer: Since the cumulative distribution function is given by

$$F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x < b \\ 1, & x \geq b \end{cases}$$

the first and third quartiles are given by

$$q_{0.25} = 0.25(b-a) + a \quad \text{and} \quad q_{0.75} = 0.75(b-a) + a.$$

15. Find the moment-generating function of the discrete random variable X that has the probability distribution given by

$$f_X(x) = 2\left(\frac{1}{3}\right)^x, \ x = 1, 2, ...$$

Use it to find the values of μ_1' and μ_2'.

Answer: The moment generating function is given by

$$M_X(t) = E(e^{Xt}) = \sum_{x=1}^{\infty} e^{xt} f_X(x) = 2\sum_{x=1}^{\infty} e^{xt}\frac{1}{3^x} = \sum_{x=1}^{\infty}\left(\frac{e^t}{3}\right)^x = \frac{2e^t}{3-e^t}$$

for $0 \le t < ln(3)$. It is a matter of calculations to see that

$$M_X'(t) = \frac{6e^t}{(3-e^t)^2} \quad (1)$$

$$M_X''(t) = \frac{6e^t(3-e^t)^2 + 12e^{2t}(3-e^t)}{(3-e^t)^4}. \quad (2)$$

Finally,

$$\mu_1' = M_X'(0) = \frac{3}{2} \quad (3)$$

$$\mu_2' = M_X''(0) = 3 \quad (4)$$

$$(5)$$

16. Derive the moment generating function of the random variable has the probability density function $f(x) = e^{-|x|}/2$ for $x \in \mathbb{R}$ and use it to find σ_X^2.

Answer: The moment generating function is given by

$$M_X(t) = E(e^{Xt}) = \int_{-\infty}^{+\infty} e^{tx} f_X(x)dx = \int_{-\infty}^{0} e^{tx}\frac{e^x}{2}dx + \int_{0}^{+\infty} e^{tx}\frac{e^{-x}}{2}dx = \frac{1}{1-t^2}.$$

Taking into account that

$$M_X'(t) = \frac{-1}{2(t+1)^2} + \frac{1}{2(t-1)^2}$$

$$M_X''(t) = \frac{1}{(t+1)^3} - \frac{1}{(t-1)^3}.$$

Therefore,

$$E(X) = M_X'(0) = 0 \quad \text{and} \quad E(X^2) = M_X''(0) = 2$$

and, consequently,

$$Var(X) = E(X^2) - (E(X))^2 = 2.$$

17. Let X and Y be two independent random variables such that the moment generating function of X is given by

$$M_X(t) = 0.2 + 0.5e^t + 0.3e^{2t}$$

and the probability function of Y is given by

$$f_Y(y) = \begin{cases} 0.3, & y = -1 \\ 0.5, & y = 1 \\ 0.2, & y = 3 \\ 0, & \text{otherwise} \end{cases}$$

a) Compute the cumulative distribution function of Y.
Solution: By definition, we can compute the CDF doing

$$F_Y(y) = P(Y \leq y) = \begin{cases} 0, & y < -1 \\ 0.3, & -1 \leq y < 1 \\ 0.8, & 1 \leq y < 3 \\ 1, & x \geq 3 \end{cases}$$

b) Compute the moment generating function of Y.
Solution: The moment generating function is, by definition,

$$\begin{aligned} M_Y(t) = E(e^{Yt}) &= \sum_{y \in D_Y} ye^{yt} \\ &= 0.3e^{-t} + 0.5e^t + 0.2e^{3t}. \end{aligned}$$

c) Compute the mode and the median of Y.
Solution:

$$\begin{aligned} mo(X) &= \arg\max_{x \in \mathbb{R}} P(X = x) \\ &= 1 \\ me(X) &= \min\{x \in \mathbb{R} : F(x) \geq 0.5\} \\ &= 1 \end{aligned}$$

d) Compute the coefficient of variation of X.
Solution: The coefficient of variation of X is, by definition $\rho_X = \sigma_X/\mu_X$. Then, one has to compute

$$\begin{aligned} E(X) = M'_X(0) = (0.5e^t + 0.6e^{2t})|_{t=0} = 1.1 \\ E(X^2) = M''_X(0) = (0.5e^t + 1.2e^{2t})|_{t=0} = 1.7. \end{aligned}$$

Then, $Var(X) = E(X^2) - (E(X))^2 = 49/100$ and, consequently, $\sigma_X = \sqrt{49/100} = 7/10$. The coefficient of variation is

$$\rho_X = \frac{\sigma_X}{\mu_X} = \frac{7/10}{11/10} = \frac{7}{11}.$$

e) Let Z be the random variable given by $Z = aY + b$. Find a and b such that $M_X(t) = M_Z(t)$.
Solution: We may notice that

$$M_Z(t) = E(e^{(aY+b)t}) = e^{bt} M_Y(at) = e^{bt}(0.3e^{-at} + 0.5e^{at} + 0.2e^{3at}).$$

Comparing $M_Z(t)$ with $M_X(t) = 0.2 + 0.5e^t + 0.3e^{2t}$ we get that

$$\begin{cases} 3a + b = 0 \\ a + b = 1 \\ b - a = 2 \end{cases} \Leftrightarrow \begin{cases} a = -1/2 \\ b = 3/2 \end{cases}$$

f) Compute the moment generating function of $W = X + Y$.
Solution: Since X and Y are two independent random variables we have

$$M_W(t) = E(e^{tW}) = E(e^{tX})E(e^{tY}) = M_X(t)M_Y(t).$$