ISEG - Lisbon School of Economics and Management

List of Exercises - Chapter 3
1^{st} Semester of 2020/2021

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1. Let X be a random variable that takes on the values 0, 1, 2, and 3 with probabilities $\frac{1}{10}, \frac{3}{10}, \frac{2}{10}, \frac{4}{10}$.
   (a) Find $E(X)$ and $E(X^2)$.
   (b) Use the results of part *(a)* to determine the value of $E[(X-\mu_X)^2]$.
   (c) Use the definition to calculate σ_X.

2. Let X be a continuous random variable and f_X its density function

$$f_X(x) = \begin{cases} 1/3, & 0 < x < 1 \\ 4/45x, & 1 < x < 4 \end{cases}.$$

   (a) Compute the expected value and the variance of X.
   (b) Compute the expected value of Y that is given by

$$Y = g(X) = \begin{cases} 0, & X < 1 \\ 1, & X \geq 1 \end{cases}.$$

   (c) Compute the expected value of $Z = 2Y - 1$.

3. The demand of a certain product, in Kg, in a random day is well represented by the random variable X with density function

$$f_X(x) = \begin{cases} 1/5, & 0 < x < 5 \\ 0, & \text{otherwise} \end{cases}.$$

   The firm that sells this product has a profit of 5 euros per Kg sold and a loss of 2 euros per Kg that is not sold.

(a) How many Kg of the product should the firm have in stock to maximize the expected profit?

(b) Assume now that X is a discrete random variable, with a probability function

$$f_X(x) = \frac{1}{6}, \quad \text{for } x = 0, 1, 2, 3, 4, 5.$$

What is the expected profit?

4. Let X be a continuous random variable and f_X its distribution. Prove that the expected value of

$$Y = \begin{cases} a, & X < 0 \\ b, & X \geq 0 \end{cases},$$

is $E(Y) = aP(X < 0) + bP(X \geq 0)$.

5. Find $E(X)$, $E(X^2)$ and σ_X^2 for the random variable X that has probability density function

$$f_X(x) = \begin{cases} \frac{x}{2} & \text{for } 0 < x < 2 \\ 0 & \text{elsewhere} \end{cases}.$$

6. Let X be a discrete random variable such that

$$f_X(x) = \begin{cases} 1/2, & x = 0 \\ 1/3, & x = 1 \\ 1/6, & x = 2 \end{cases}$$

Compute γ_1.

7. Let X be a continuous random variable such that

$$f_X(x) = \begin{cases} x, & 0 < x < 1 \\ 1/2, & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Compute γ_2.

8. Let X be a random variable that has probability density function

$$f(x) = \begin{cases} x/2 & \text{for } 0 < x \leq 1 \\ 1/2 & \text{for } 1 < x \leq 2 \\ (3 - x)/2 & \text{for } 2 < x < 3 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find $E(X)$, the median and the mode of X.

(b) Find $E(X^2)$.

(c) Use the results of part (a) and (b) to determine $E(X^2 - 5X + 3)$.

(d) Compute the standard deviation.

9. Find the expected value, the median and the mode of the discrete random variable X having the probability distribution $f_X(x) = |x-2|/7$, $x = -1, 0, 1, 3$.

10. Find the expected value, the median and the mode, of the random variable Y whose probability density is given by

$$f_Y(y) = \begin{cases} (y+1)/8 & \text{for } 2 \le y \le 4 \\ 0 & \text{elsewhere} \end{cases}$$

11. Let X be a random variable that has the probability function $f_X(x) = 1/2$ for for $x = -2$ and $x = 2$.

(a) Find $E(X)$, $E(X^2)$ and σ_X^2.

(b) Calculate the the mode and median.

(c) Calculate first and third quartiles.

(d) Compute the standard deviation.

(e) Compute $Var(2X - 2)$

12. Let X be a random variable that has probability density function

$$f_X(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2-x & \text{for } 1 \le x < 2 \\ 0 & \text{elsewhere} \end{cases}.$$

(a) Find the expected value, the median and the mode of the random variable X.

(b) Compute the variance of $g(X) = 2X + 3$.

13. Let X be a random variable that has probability density function

$$f(x) = \begin{cases} \frac{1}{x \log(3)} & \text{for } 1 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}.$$

(a) Find $E(X)$, the median and the mode of X.

(b) Find $E(X^2)$ and $E(X^3)$.

(c) Use the results of part (a) and (b) to determine $E(X^3 + 2X^2 - 3X + 1)$.

14. Let X be a random variable such that

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0 \end{cases}$$

(a) Find the moment generating function of X.

(b) Calculate the first and third quantiles.

15. Find the moment-generating function of the discrete random variable X that has the probability distribution given by

$$f(x) = 2\left(\frac{1}{3}\right)^x, \; x = 1, 2, ...$$

Use it to find the values of μ_1' and μ_2'.

16. Derive the moment generating function of the random variable has the probability density function $f(x) = e^{-|x|}/2$ for $x \in \mathbb{R}$ and use it to find σ_X^2.

17. Let X and Y be two independent random variables such that the moment generating function of X is given by

$$M_X(t) = 0.2 + 0.5e^t + 0.3e^{2t}$$

and the probability function of Y is given by

$$f_Y(y) = \begin{cases} 0.3, & y = -1 \\ 0.5, & y = 1 \\ 0.2, & y = 3 \\ 0, & \text{otherwise} \end{cases}$$

a) Compute the cumulative distribution function of Y.

b) Compute the moment generating function of Y.

c) Compute the mode and the median of Y.

d) Compute the coefficient of variation of X.

e) Let Z be the random variable given by $Z = aY + b$. Find a and b such that $M_X(t) = M_Z(t)$.

f) Compute the moment generating function of $W = X + Y$.