ISEG - Lisbon School of Economics and Management

List of Exercises - Chapter 2
2^{nd} Semester of 2019/2020

October 21, 2020

1. For each of the following, determine whether the given values can serve as the values of a probability function of a random variable with the range $x = 1$, 2, 3, and 4:

   a) $f(1) = 0.25$, $f(2) = 0.75$, $f(3) = 0.25$, and $f(4) = -0.25$;
   Answer: f is not a probability function because $f(4) < 0$, which is not allowed.

   b) $f(1) = 0.15$, $f(2) = 0.27$, $f(3) = 0.29$, and $f(4) = 0.29$;
   Answer: f can be a probability function because $f(x) > 0$ for all x in the range specified and
   $$\sum_{x=1}^{4} f(x) = 1.$$

   c) $f(1) = 1/19$, $f(2) = 10/19$, $f(3) = 2/19$, and $f(4) = 5/19$.
   Answer: f is not a probability function because
   $$\sum_{x=1}^{4} f(x) = \frac{18}{19} \neq 1$$

2. Verify that $f(x) = 2x/[k(k+1)]$ for $x = 1, 2, 3, ..., k$ can serve as the probability function of a random variable with the given range.

3. For what values of k can $f(x) = (1-k)k^x$ serve as the values of the probability function of a random variable with the countably infinite range $x = 0, 1, 2, ...$?

   Answer: $k \in [0, 1[$

4. Show that $f(x) = 1/x$ cannot serve as the values of the probability function of a random variable with the countably infinite range $x = 1, 2, 3, ...$

5. For each of the following, determine whether the given values can serve as the values of a cumulative distribution function of a random variable with the range $x = 1, 2, 3$, and 4:

   (a) $F(1) = 0.3$, $F(2) = 0.5$, $F(3) = 0.8$, and $F(4) = 1.2$;

   Answer: F is not a cumulative distribution function because $F(4) > 1$, which is not allowed because a CDF has to verify $0 \leq F \leq 1$.

   (b) $F(1) = 0.5$, $F(2) = 0.4$, $F(3) = 0.7$, and $F(4) = 1.0$;

   Answer: F is not a cumulative distribution function because $F(2) < F(1)$, which means that F decreases, which is not allowed.

   (c) $F(1) = 0.25$, $F(2) = 0.61$, $F(3) = 0.83$, and $F(4) = 1.0$.

   Answer: F can be a cumulative distribution function because it seems that F is non-decreasing, $0 \leq F \leq 1$ and F is right continuous.

6. If X has the cumulative distribution function

$$F_X(x) = \begin{cases} 0 & \text{for } x < 1 \\ 1/3 & \text{for } 1 \leq x < 4 \\ 1/2 & \text{for } 4 \leq x < 6 \\ 5/6 & \text{for } 6 \leq x < 10 \\ 1 & \text{for } x \geq 10 \end{cases}$$

   find

   (a) $P(2 < X \leq 6)$;

   Answer: $P(2 < X \leq 6) = F_X(6) - F_X(2) = \frac{1}{2}$

   (b) $P(X = 4)$;

   Answer: $P(X = 4) = F_X(4) - F_X(4^-) = \frac{1}{6}$.

   (c) the probability function of X.

   Answer:

$$f_X(x) = \begin{cases} \frac{1}{3}, & x = 1 \\ \frac{1}{6}, & x = 4 \\ \frac{1}{3}, & x = 6 \\ \frac{1}{6}, & x = 10 \\ 0, & \text{otherwise} \end{cases}$$

7. Find the cumulative distribution function of the random variable that has the probability function $f(x) = x/15$ for $x = 1, 2, 3, 4, 5$.

Answer: The cumulative distribution function is

$$F_X(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{15}, & 1 < x \leq 2 \\ \frac{3}{15}, & 2 < x \leq 3 \\ \frac{6}{15}, & 3 < x \leq 4 \\ \frac{10}{15}, & 4 < x \leq 5 \\ 1, & x \geq 5 \end{cases}.$$

8. To make a study about the quality of public transports in a certain city, the Mayor wants to know how many people arrive at a bus stop to catch a bus between two consecutive bus arrivals. Let X be a random variable that provides this information, with the following probability function:

x	0	1	2	3	4	5	6 or more
$P(X = x)$	0.1	0.15	0.20	0.25	a	b	0.05

Find a and b such that

a) $P(X \geq 5) = 0.15$;
Answer: $a = 0.15$ and $b = 0.1$

b) $P(X \in \{1, 4\}) = 0.35$;
Answer: $a = 0.2$ and $b = 0.05$

c) $F_X(4) = 0.8$.
Answer: $a = 0.1$ and $b = 0.15$

9. The probability density of the continuous random variable X is given by

$$f_X(x) = \begin{cases} 1/5 & 2 < x < 7 \\ 0 & elsewhere \end{cases}$$

(a) Draw its graph and verify that the total area under the curve (above the x-axis) is equal to 1.

(b) Find $P(3 < X < 5)$.
Answer: $P(3 < X < 5) = \frac{2}{5}$

10. Let $f_X(x) = e^{-x}$ for $0 < x < +\infty$.

(a) Show that $f_X(x)$ represents a probability density function.

(b) Sketch a graph of this function and indicate the area associated with the probability that $X > 1$.

(c) Calculate the probability that $X > 1$.
Answer: $P(X > 1) = e^{-1}$.

11. Let $f_X(x) = 3x^2$ for $0 < x < 1$.

(a) Show that $f_X(x)$ represents a density function.

(b) Sketch a graph of this function, and indicate the area associated with the probability that $0.1 < X < 0.5$.

(c) Calculate the probability that $0.1 < X < 0.5$.
Answer: $P(0.1 < X < 0.5) = 0.124$.

12. The probability density function of the random variable X is given by

$$f_X(x) = \begin{cases} \frac{c}{\sqrt{x}} & 0 < x < 4 \\ 0 & elsewhere \end{cases}$$

Find

(a) the value of c;
Answer: f_X is a density probability function, thus it satisfies

$$f_X(x) \geq 0, \ \forall x \in \mathbb{R} \quad \text{and} \quad \int_{-\infty}^{+\infty} f_X(x)dx = 1.$$

Form the first condition, we get that $c \geq 0$ and from the second we obtain

$$\int_{-\infty}^{+\infty} f_X(x)dx = 1 \Leftrightarrow \int_0^4 \frac{c}{\sqrt{x}}dx = 1 \Leftrightarrow c = \frac{1}{4}.$$

(b) $P(X < 14)$ and $P(X > 1)$.
Answer:

$$P(X < 14) = \int_{-\infty}^{14} f_X(x)dx = \int_0^4 f_X(x)dx = 1, \quad (1)$$

$$P(X > 1) = \int_1^{+\infty} f_X(x)dx = \int_1^4 \frac{1}{4\sqrt{x}}dx = \frac{1}{2}. \quad (2)$$

13. The probability density of the random variable Z is given by

$$f_Z(z) = \begin{cases} kze^{-z^2} & z > 0 \\ 0 & z \leq 0 \end{cases}$$

Find k.

Answer: $k = 2$.

14. Find the cumulative distribution function of the random variable X whose probability density is given by

$$f_X(x) = \begin{cases} \frac{1}{3}, & 0 < x < 1 \\ \frac{1}{3}, & 2 < x < 4 \\ 0, & \text{elsewhere.} \end{cases}$$

Also sketch the graphs of the probability density and distribution functions.

Answer: The cumulative distribution function is the $F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(u)du$. Therefore,

$$F_X(x) = \begin{cases} 0, & x \leq 0 \\ \frac{x}{3}, & 0 \leq x < 1 \\ \frac{1}{3}, & 1 \leq x < 2 \\ \frac{x}{3} - \frac{1}{3}, & 2 \leq x < 4 \\ 1, & x \geq 4 \end{cases}.$$

15. Find the cumulative distribution function of the random variable X whose probability density is given by

$$f_X(x) = \begin{cases} \frac{x}{2}, & 0 < x \leq 1 \\ \frac{1}{2}, & 1 < x \leq 2 \\ \frac{3-x}{2}, & 2 < x < 3 \\ 0, & \text{elsewhere.} \end{cases}$$

Also sketch the graphs of these probability density and distribution functions.

Answer:

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{4}, & 0 \leq x < 1 \\ \frac{x}{2} - \frac{1}{4}, & 1 \leq x < 2 \\ -\frac{5}{4} + \frac{3}{2}x - \frac{x^2}{4}, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

16. The cumulative distribution function of the random variable X is given by

$$F_X(x) = \begin{cases} 1-(1+x)e^{-x} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find $P(X \le 2)$, $P(1 < X < 3)$, and $P(X > 4)$.

Answer:

$$\begin{aligned} &P(X \le 2) = F_X(2) = 1 - 3e^{-2} \\ &P(1 < X < 3) = P(1 < X \le 3) = F_X(3) - F_X(1) = 2e^{-1} - 4e^{-3} \\ &P(X > 4) = 1 - F_X(4) = 5e^{-4} \end{aligned}$$

17. The distribution function of the random variable X is given by

$$F_X(x) = \begin{cases} 0, & x \le -1 \\ \frac{x+1}{2}, & -1 \le x < 1 \\ 1, & x \ge 1. \end{cases}$$

Find

a) the probability function f_X;
Answer: It is known that $f_X(x) = F'_X(x)$ almost everywhere. Therefore,

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1 \\ 0, & otherwise \end{cases}.$$

b) $P(-\frac{1}{2} < X \le -\frac{1}{2})$;
Answer:

$$P(-\frac{1}{2} < X \le -\frac{1}{2}) = F_X\left(-\frac{1}{2}\right) - F_X\left(-\frac{1}{2}\right) = 0.$$

c) $P(2 < X < 3)$.
Answer:

$$P(2 < X < 3) = P(2 < X \le 3) = F_X(3) - F_X(2) = 0$$

18. The cumulative distribution function of the random variable Z is given by

$$F_Z(z) = \begin{cases} 0 & \text{for } z < -2 \\ \frac{z+4}{8} & \text{for } -2 \le z < 2 \\ 1 & \text{for } z \ge 2 \end{cases}$$

a) Sketch the graph of the distribution function F_Z;

b) Is Z a continuous random variable? Why?
Answer: No, because F is not continuous.

c) Compute $P(Z = -2)$, $P(Z = 2)$, $P(-2 < Z < 1)$, and $P(0 \leq Z \leq 2)$.
Answer: $P(Z = -2) = 1/4$, $P(Z = 2) = 1/4$, $P(-2 < Z < 1) = 3/8$, $P(0 \leq Z \leq 2) = 1/2$

19. Let X be a random variable with cumulative distribution function given by

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{6}x, & 0 \leq x < 1 \\ \frac{1}{3}, & 1 \leq x < 2 \\ 1, & x \geq 2 \end{cases}.$$

a) Prove that X is a mixed random variable;
Answer: X is a mixed random variable if (a) $D_X \neq \emptyset$ and $P(X \in D_X) < 1$ and (b) there is $\lambda \in (0,1)$, X_1 a discrete random variable and X_2 a continuous random variable such that $F_X(x) = \lambda F_{X_1}(x) + (1-\lambda)F_{X_2}(x)$.
(a) $D_X = \{1, 2\}$ and

$$\begin{aligned} P(X \in D_X) &= P(X = 1) + P(X = 2) \\ &= F_X(1) - F_X(1^-) + F_X(2) - F_X(2^-) \\ &= \left(\frac{1}{3} - \frac{1}{6}\right) + \left(1 - \frac{1}{3}\right) = \frac{5}{6} < 1 \end{aligned}$$

(b)
We notice that

$$\begin{aligned} P(X = x) &= F_X(x) - F_X(x^-) = \lambda(F_{X_1}(x) - F_{X_1}(x^-)) + (1-\lambda)(F_{X_2}(x) - F_{X_2}(x^-)) \\ &= \lambda(F_{X_1}(x) - F_{X_1}(x^-)) = \lambda P(X_1 = x). \end{aligned}$$

Therefore,

$$P(X = 1) + P(X = 2) = \lambda(P(X_1 = 1) + P(X_2 = 1)) = \lambda,$$

because, $P(X_1 = 1) + P(X_2 = 2) = 1$. ($D_{X_1} = \{1, 2\}$ and X_1 is discrete random variable.) It is a matter os calculations to see that

$$\begin{aligned} F_X(x) &= \frac{5}{6} \times \begin{cases} 0, & x < 1 \\ \frac{1}{5}, & 1 \leq x < 2 \\ 1, & x \geq 2 \end{cases} + \frac{1}{6} \times \begin{cases} 0, & x < 0 \\ x, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases} \\ &= \frac{5}{6}F_{X_1}(x) + \frac{1}{6}F_{X_2}(x) \end{aligned}$$

b) Calculate the probabilities: $P(X < 1/2)$, $P(X < 3/2)$, $P(1/2 < X < 2)$, $P(X = 1)$, $P(X > 1)$, $P(X = 2)$.

Answer:

$$
\begin{aligned}
P(X < 1/2) &= F_X(1/2) = \frac{1}{12} \\
P(X < 3/2) &= F_X(3/2) = \frac{1}{3} \\
P(1/2 < X < 2) &= P(1/2 < X \leq 2) - P(X = 2) = F_X(2) - F_X(1/2) - P(X = 2) = \frac{1}{4} \\
P(X = 1) &= F_X(1) - F_X(1^-) = \frac{1}{3} - \frac{1}{6} = \frac{1}{6} \\
P(X > 1) &= 1 - F_X(1) = 1 - \frac{1}{3} = \frac{2}{3} \\
P(X = 2) &= F_X(2) - F_X(2^-) = 1 - \frac{1}{3} = \frac{2}{3}.
\end{aligned}
$$

20. If the probability density of X is given by

$$f_X(x) = \begin{cases} \frac{x}{2}, & 0 < x < 2 \\ 0, & \text{elsewhere} \end{cases}.$$

a) Compute the cumulative distribution function of $Y = X^3$.

Solution: We start by calculating the Cumulative distribution function.

$$F_Y(y) = P(Y \leq y) = P(X^3 \leq y) = P(X \leq \sqrt[3]{y}) = F_X(\sqrt[3]{y}).$$

It is a matter of computations to see that

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{4}, & 0 \leq x < 2 \\ 1, & x \geq 2 \end{cases}$$

Therefore, we get the following result

$$F_Y(y) = F_X(\sqrt[3]{y}) = \begin{cases} 0, & y < 0 \\ \frac{y^{\frac{2}{3}}}{4}, & 0 \leq y < 8 \\ 1, & y \geq 8 \end{cases}.$$

b) Compute the probability density function of $Y = X^3$.

Solution: We know that $f_Y(y) = F_Y(y)$ almost everywhere. Therefore, computing the derivative of F_Y we get

$$f_Y(y) = \begin{cases} \frac{1}{6}y^{-\frac{1}{3}}, & 0 < y < 8 \\ 0, & \text{otherwise} \end{cases}$$

21. Let X be a discrete random variable such that

$$P(X = x) = \frac{1}{7}, \quad \text{with } x \in \{-3, -2, -1, 0, 1, 2, 3\}.$$

Determine the probability function of $Y = X^2 - 3X$.

Answer: Firstly, we notice that which means that

X	-3	-2	-1	0	1	2	3
$X^2 - 3X$	18	10	4	0	-2	-2	0

$$\begin{aligned}
P(Y = -2) &= P(X = 1) + P(X = 2) = \frac{2}{7} \\
P(Y = 0) &= P(X = 0) + P(X = 3) = \frac{2}{7} \\
P(Y = 4) &= P(X = -1) = \frac{1}{7} \\
P(Y = 10) &= P(X = -2) = \frac{1}{7} \\
P(Y = 18) &= P(X = -3) = \frac{1}{7} \\
P(Y = y) &= 0, \quad \text{for all } x \notin \{-2, 0, 4, 10, 18\}.
\end{aligned}$$

22. If the probability density of X is given by

$$f_X(x) = \begin{cases} \frac{3x^2}{2}, & -1 < x < 1 \\ 0, & \text{elsewhere} \end{cases}.$$

Determine the density function of $Y = |X|$ and $Z = X^2$.

Answer:

$$f_Y(y) = \begin{cases} 3y^2, & 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases} \quad \text{and} \quad f_Z(z) = \begin{cases} \frac{3z^{1/2}}{2}, & 0 < z < 1 \\ 0, & \text{elsewhere} \end{cases}.$$