ISEG - Lisbon School of Economics and Management

List of Exercises - Chapter 2
1^{st} Semester of 2020/2021

March 3, 2021

1. For each of the following, determine whether the given values can serve as the values of a probability function of a random variable with the range $x = 1$, 2, 3, and 4:

   a) $f(1) = 0.25$, $f(2) = 0.75$, $f(3) = 0.25$, and $f(4) = -0.25$;
   b) $f(1) = 0.15$, $f(2) = 0.27$, $f(3) = 0.29$, and $f(4) = 0.29$;
   c) $f(1) = 1/19$, $f(2) = 10/19$, $f(3) = 2/19$, and $f(4) = 5/19$.

2. Verify that $f(x) = 2x/[k(k+1)]$ for $x = 1, 2, 3, ..., k$ can serve as the probability function of a random variable with the given range.

3. For what values of k can $f(x) = (1-k)k^x$ serve as the values of the probability function of a random variable with the countably infinite range $x = 0, 1, 2, ...$?

4. Show that $f(x) = 1/x$ cannot serve as the values of the probability function of a random variable with the countably infinite range $x = 1, 2, 3, ...$

5. For each of the following, determine whether the given values can serve as the values of a cumulative distribution function of a random variable with the range $x = 1, 2, 3$, and 4:

   (a) $F(1) = 0.3$, $F(2) = 0.5$, $F(3) = 0.8$, and $F(4) = 1.2$;
   (b) $F(1) = 0.5$, $F(2) = 0.4$, $F(3) = 0.7$, and $F(4) = 1.0$;
   (c) $F(1) = 0.25$, $F(2) = 0.61$, $F(3) = 0.83$, and $F(4) = 1.0$.

6. If X has the cumulative distribution function

$$F_X(x) = \begin{cases} 0 & \text{for } x < 1 \\ 1/3 & \text{for } 1 \leq x < 4 \\ 1/2 & \text{for } 4 \leq x < 6 \\ 5/6 & \text{for } 6 \leq x < 10 \\ 1 & \text{for } x \geq 10 \end{cases}$$

find

(a) $P(2 < X \leq 6)$;
(b) $P(X = 4)$;
(c) the probability function of X.

7. Find the cumulative distribution function of the random variable that has the probability function $f(x) = x/15$ for $x = 1, 2, 3, 4, 5$.

8. To make a study about the quality of public transports in a certain city, the Mayor wants to know how many people arrive at a bus stop to catch a bus between two consecutive bus arrivals. Let X be a random variable that provides this information, with the following probability function:

x	0	1	2	3	4	5	6 or more
$P(X = x)$	0.1	0.15	0.20	0.25	a	b	0.05

Find a and b such that

a) $P(X \geq 5) = 0.15$;
b) $P(X \in \{1, 4\}) = 0.35$;
c) $F_X(4) = 0.8$.

9. The probability density of the continuous random variable X is given by

$$f_X(x) = \begin{cases} 1/5 & 2 < x < 7 \\ 0 & \textit{elsewhere} \end{cases}$$

(a) Draw its graph and verify that the total area under the curve (above the x-axis) is equal to 1.
(b) Find $P(3 < X < 5)$.

10. Let $f_X(x) = e^{-x}$ for $0 < x < +\infty$.

(a) Show that $f_X(x)$ represents a probability density function.
(b) Sketch a graph of this function and indicate the area associated with the probability that $X > 1$.
(c) Calculate the probability that $X > 1$.

11. Let $f_X(x) = 3x^2$ for $0 < x < 1$.

(a) Show that $f_X(x)$ represents a density function.

(b) Sketch a graph of this function, and indicate the area associated with the probability that $0.1 < X < 0.5$.

(c) Calculate the probability that $0.1 < X < 0.5$.

12. The probability density function of the random variable X is given by

$$f_X(x) = \begin{cases} \frac{c}{\sqrt{x}} & 0 < x < 4 \\ 0 & elsewhere \end{cases}$$

Find

(a) the value of c;

(b) $P(X < 14)$ and $P(X > 1)$.

13. The probability density of the random variable Z is given by

$$f_Z(z) = \begin{cases} kze^{-z^2} & z > 0 \\ 0 & z \leq 0 \end{cases}$$

Find k.

14. Find the cumulative distribution function of the random variable X whose probability density is given by

$$f_X(x) = \begin{cases} \frac{1}{3}, & 0 < x < 1 \\ \frac{1}{3}, & 2 < x < 4 \\ 0, & \text{elsewhere.} \end{cases}$$

Also sketch the graphs of the probability density and distribution functions.

15. Find the cumulative distribution function of the random variable X whose probability density is given by

$$f_X(x) = \begin{cases} \frac{x}{2}, & 0 < x \leq 1 \\ \frac{1}{2}, & 1 < x \leq 2 \\ \frac{3-x}{2}, & 2 < x < 3 \\ 0, & \text{elsewhere.} \end{cases}$$

Also sketch the graphs of these probability density and distribution functions.

16. The cumulative distribution function of the random variable X is given by

$$F_X(x) = \begin{cases} 1 - (1+x)e^{-x} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find $P(X \leq 2)$, $P(1 < X < 3)$, and $P(X > 4)$.

17. The distribution function of the random variable X is given by

$$F_X(x) = \begin{cases} 0, & x \leq -1 \\ \frac{x+1}{2}, & -1 \leq x < 1 \\ 1, & x \geq 1. \end{cases}$$

Find

a) the probability function f_X;
b) $P(-\frac{1}{2} < X \leq -\frac{1}{2})$;
c) $P(2 < X < 3)$.

18. The cumulative distribution function of the random variable Z is given by

$$F_Z(z) = \begin{cases} 0 & \text{for } z < -2 \\ \frac{z+4}{8} & \text{for } -2 \leq z < 2 \\ 1 & \text{for } z \geq 2 \end{cases}$$

a) Sketch the graph of the distribution function F_Z;
b) Is Z a continuous random variable? Why?
c) Compute $P(Z = -2)$, $P(Z = 2)$, $P(-2 < Z < 1)$, and $P(0 \leq Z \leq 2)$.

19. Let X be a random variable with cumulative distribution function given by

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{6}x, & 0 \leq x < 1 \\ \frac{1}{3}, & 1 \leq x < 2 \\ 1, & x \geq 2 \end{cases}.$$

a) Prove that X is a mixed random variable;
b) Calculate the probabilities: $P(X < 1/2)$, $P(X < 3/2)$, $P(1/2 < X < 2)$, $P(X = 1)$, $P(X > 1)$, $P(X = 2)$.

20. If the probability density of X is given by

$$f_X(x) = \begin{cases} \frac{x}{2}, & 0 < x < 2 \\ 0, & \text{elsewhere} \end{cases}.$$

a) Compute the cumulative distribution function of $Y = X^3$.
b) Compute the probability density function of $Y = X^3$.

21. Let X be a discrete random variable such that

$$P(X = x) = \frac{1}{7}, \quad \text{with } x \in \{-3, -2, -1, 0, 1, 2, 3\}.$$

Determine the probability function of $Y = X^2 - 3X$.

22. If the probability density of X is given by

$$f_X(x) = \begin{cases} \frac{3x^2}{2}, & -1 < x < 1 \\ 0, & \text{elsewhere} \end{cases}.$$

Determine the cumulative distribution function of $Y = |X|$ and $Z = X^2$.

23. A company has received €1 million to invest in one of two projects. With probability 1/3 the firm invests in project 1, and, with probability 2/3, the firm invests in project 2. Let X_1 be a discrete random variable that represents the return of project 1, in millions of €, and X_2 a continuous random variable that represents the return , of project 2, in millions of €. The cumulative distribution function of X_1 is given by

$$F_{X_1}(x) = \begin{cases} 0, & x < 0 \\ 1/3, & 0 \le x < 1 \\ 1/2, & 1 \le x < 2 \\ 7/10, & 2 \le x < 3 \\ 1, & x \ge 3 \end{cases},$$

and the probability density function of X_2 is given by

$$f_{X_2}(x) = \begin{cases} 1/3, & 0 < x < 1 \\ \frac{4}{45}x, & 1 \le x < 4 \\ 0, & \text{otherwise} \end{cases}$$

(a) Compute the probability function of X_1.

(b) Compute the cumulative distribution function of X_2

(c) If the firm invests in project 1, what is the probability that it gets back more than 2 million? Compute the same probability in case the firm invests in project 2.

(d) Compute $P(X_2 > 1 | X_2 < 3)$.

(e) Let X be the amount, in millions of €, received by the company after its investment.

i. Find the cumulative distribution function of X. Classify the random variable.

ii. Compute the probability that the firm receives at least 3 million.

iii. Compute the probability that the firm has a negative profit (profit = amount received minus amount invested).