ISEG - Lisbon School of Economics and Management

List of Exercises - Chapter 1
1^{st} Semester of 2020/2021

October 20, 2020

1. 1. A box contains 5 balls of which 2 are black. The black balls are numbered 1 and 2, the others 3 to 5. Two balls are randomly taken out one after the other without replacement. The numbers on the two balls are observed.

   a) List all the elements of the sample space associated to this random experiment.

   Answer: B_iB_j="Ball i is taken in the first try and ball j is taken in the second try"

   $$S = \{B_iB_j \ : i \neq j \text{ and } i, j = 1, 2, \cdots, 5\}$$

   b) From the sample space, define the following events:

   A_1 - The first ball observed is black;
   A_2 - The second ball observed is black;
   A_3 - The two balls taken out are black;
   A_4 - At least one of the two balls are black;
   A_5 - Exactly one of the two balls are black;
   A_6 - The sum of the numbers in the two balls is greater than seven.

   Answer:

   $$\begin{aligned}
   &A_1 = \{B_iB_j \ : i = 1, 2, \ j = 1, 2, \cdots, 5, \text{ and } i \neq j\} \\
   &A_2 = \{B_iB_j \ : j = 1, 2, \ i = 1, 2, \cdots, 5, \text{ and } i \neq j\} \\
   &A_3 = \{B_1B_2, B_2B_1\} \\
   &A_4 = A_1 \cup A_2 \\
   &A_5 = \{B_iB_j \ : (i = 1, 2 \text{ and } j = 3, 4, 5) \text{ or } (j = 1, 2 \text{ and } i = 3, 4, 5)\} \\
   &A_6 = \{B_3B_5, B_4B_5, B_5B_3, B_5B_4\}
   \end{aligned}$$

2. Two light bulbs will be kept on until both turned off. The life time of both bulbs are registered. None of the light bulbs have a life time greater than 1600 hours. Represent the sample space and the following events:

   A - None of the light bulbs has a life time longer than 1000 hours;

   B - Just one of the bulbs has a life time greater than 1000 hours;

   C - The life time of one of the bulbs doubles the life time of the other;

   D - The sum of the life times of the two bulbs is greater than 2000 hours.

   Answer:

   - Sample space: $S = \{(l_1, l_2) \in \mathbb{R}^2 : 0 \leq l_1 \leq 1600, 0 \leq l_2 \leq 1600\}$;
   - $A = \{(l_1, l_2) \in \mathbb{R}^2 : 0 \leq l_1 \leq 1000, 0 \leq l_2 \leq 1000\}$;
   - $B = \{(l_1, l_2) \in \mathbb{R}^2 : (0 \leq l_1 \leq 1000, 1000 < l_2 \leq 1600) \vee (1000 < l_1 \leq 1600, 0 \leq l_2 \leq 1000)\}$;
   - $C = \{(l_1, l_2) \in \mathbb{R}^2 : (l_1 = 2l_2 \vee l2 = 2l_1) \wedge (l_1 \leq 1600, 0 \leq l_2 \leq 1600)\}$;
   - $D = \{(l_1, l_2) \in \mathbb{R}^2 : l_1 + l_2 > 2000 \wedge (l_1 \leq 1600, 0 \leq l_2 \leq 1600)\}$

3. Define and classify the sample spaces associated with the following random experiments:

   a) Observation of the number of spots when a six face die is thrown.
   Answer: $S = \{1, 2, 3, 4, 5, 6\}$, S is discrete (and finite).

   b) A coin is tossed and the face-up is observed.
   Answer: $S = \{H, T\}$, S is discrete (and finite).

   c) A die is thrown followed by the toss of a coin.
   Answer: $S = \{(i, \alpha) : i = 1, 2, \cdots, 6 \text{ and } \alpha = H, T\}$, S is discrete (and finite).

   d) A coin is tossed until a head is obtained.
   Answer: $S = \{H, (T, H), (T, T, H), (T, T, T, H), (T, T, T, T, H), \cdots\}$, S is discrete (and countably infinite).

4. Let A and B be two events in the sample space S. Verify the following equalities:

   a) $(A \cap B) \cup (A \cap \overline{B}) = A$;
   Answer:

$$(A \cap B) \cup (A \cap \overline{B}) = \underbrace{\big(A \cup (A \cap \overline{B})\big)}_{A} \cap \big(B \cup (A \cap \overline{B})\big)$$

$$= A \cap \left((B \cup A) \cap \underbrace{(B \cup \overline{B})}_{S} \right) = A \cap (A \cup B) = A$$

b) $\overline{(A \cap \overline{B}) \cup (B \cap \overline{A})} = \overline{A \cup B} \cup (A \cap B)$;

Answer:

$$\begin{aligned}
\overline{(A \cap \overline{B}) \cup (B \cap \overline{A})} &= \overline{A \cap \overline{B}} \cap \overline{B \cap \overline{A}} = (\overline{A} \cup B) \cap (\overline{B} \cup A) \\
&= ((\overline{A} \cup B) \cap \overline{B}) \cup ((\overline{A} \cup B) \cap A)) \\
&= (\overline{A} \cap \overline{B}) \cup (B \cap \overline{B}) \cup (\overline{A} \cap A) \cup (B \cap A) \\
&= \overline{A \cup B} \cup (A \cap B),
\end{aligned}$$

where, in the first, second and last equalities we are using De Morgan's Laws.

c) $A \cap B = \overline{\overline{A} \cup \overline{B}}$;

Answer:

$$A \cap B = \overline{\overline{A \cap B}} = \overline{\overline{A} \cup \overline{B}}.$$

d) $(A \cap \overline{B}) \cup (B \cap \overline{A}) = (A \cup B) \setminus (A \cap B)$;

Answer: Firstly, note that

$$(A \cap \overline{B}) \cup (B \cap \overline{A}) = \overline{\overline{(A \cap \overline{B}) \cup (B \cap \overline{A})}}.$$

By using the equality in question b), we have

$$\overline{(A \cap \overline{B}) \cup (B \cap \overline{A})} = \overline{A \cup B} \cup (A \cap B) \Leftrightarrow \overline{\overline{(A \cap \overline{B}) \cup (B \cap \overline{A})}} = \overline{\overline{A \cup B} \cup (A \cap B)}.$$

Then,

$$\begin{aligned}
(A \cap \overline{B}) \cup (B \cap \overline{A}) &= \overline{\overline{A \cup B} \cup (A \cap B)} = (A \cup B) \cap \overline{A \cap B} \\
&= (A \cup B) \setminus (A \cap B)
\end{aligned}$$

Display these events in a Venn Diagram.

5. Let A, B and C be three events in the sample space S. Verify which of the following is true. Justify your answer.

a) $(A \cap B) \cup C = A \cap (B \cup C)$;

Answer: False:

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C).$$

b) $(A \cap B) \cap C = A \cap (B \cap C)$;

Answer: True: Associative Property

c) $S \setminus (A \cap B) = \overline{A} \cup \overline{B}$.

Answer: True:

$$S \setminus (A \cap B) = \overline{A \cap B} = \overline{A} \cup \overline{B}.$$

where we have used De Morgan's Laws.

6. Let A and B be two events in the sample space S. Show that

a) $P(A \cap B) \geq P(A) + P(B) - 1$;

b) $P(A \cap B) \leq P(A) + P(B)$;

Answer:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \Leftrightarrow P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

Since $0 \leq P(A \cup B) \leq 1 \Leftrightarrow -1 \leq -P(A \cup B) \leq 0$, we get

$$\begin{aligned} P(A) + P(B) - P(A \cup B) &\geq P(A) + P(B) - 1 \\ P(A) + P(B) - P(A \cup B) &\leq P(A) + P(B). \end{aligned}$$

c) $A \subset B \Rightarrow P(A \cap B) = P(A)$;

Answer: $A \subset B \Rightarrow A \cap B = A \Rightarrow P(A \cap B) = P(A)$.

d) $P(A \cap B) \leq \min(P(A), P(B))$

Answer: Since $A \cap B \subset A$ and $A \cap B \subset B$, we have by question c) that

$$\begin{aligned} P(A \cap B) &\leq P(A) \\ P(A \cap B) &\leq P(B) \end{aligned}$$

Then, $P(A \cap B) \leq \min(P(A), P(B))$.

7. Let A and B be two events in the sample space S. Prove that

$$P((A \cap \overline{B}) \cup (B \cap \overline{A})) = P(A) + P(B) - 2P(A \cap B).$$

Answer: From problem 4d) we have that

$$(A \cap \overline{B}) \cup (B \cap \overline{A}) = (A \cup B) \setminus (A \cap B).$$

Therefore,

$$\begin{aligned} P((A \cap \overline{B}) \cup (B \cap \overline{A})) &= P((A \cup B) \setminus (A \cap B)) = P(A \cup B) - P((A \cup B) \cap (A \cap B)) \\ &= P(A \cup B) - P(A \cap B) = P(A) + P(B) - 2P(A \cap B) \end{aligned}$$

8. In the manufacturing of a product, two errors may occur. The error A occurs 10% of the time and the error B occurs 5% of the time. Additionally, in 12% of the time, either A or B occur. Compute the probability that a random unit of the product has

   a) both errors;
   b) only A;
   c) only B;
   d) no error.

   Answer: Let A (resp., B) be the event that occurs when error A occurs (resp., B) happens. From the problem we get the following information:

   $$P(A) = 0.1, \quad P(B) = 0.05 \quad \text{and} \quad P(A \cup B) = 0.12$$

   a) $P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.03$.
   b) $P(A \setminus B) = P(A) - P(A \cap B) = 0.07$.
   c) $P(B \setminus A) = P(B) - P(A \cap B) = 0.02$.
   d) $P(\overline{A \cup B}) = 1 - P(A \cup B) = 0.88$.

9. An electronic system is composed of two sub-systems, A and B. From previous experiments it is known that : the probability of A failing is 0.2, the probability that only B fails is 0.15 and the probability that A and B fail simultaneously is 0.15. Evaluate the probability that:

   a) B fails.
      Answer: $P(B) = 0.3$
   b) Only A fails.
      Answer: $P(A \setminus B) = 0.05$
   c) Either A or B fails.
      Answer: $P(A \cup B) = 0.35$
   d) Neither A nor B fails.
      Answer: $P(\overline{A} \cap \overline{B}) = 0.65$
   e) A and B don't fail simultaneously.
      Answer: $P(\overline{A \cap B}) = 0.85$

10. In a College, 70% of the students have a computer at home, 40% have a PC and 30% have both. If a student is randomly chosen, evaluate the probability of the student:

a) Has at least one of the two types of computers;
Answer: From the problem we have the events:

$$HC = \text{"A student has a computer at home"} \quad (1)$$
$$PC = \text{"A student has a portable computer at home"}, \quad (2)$$

and the probabilities

$$P(HC) = 0.7 \quad P(PC) = 0.4 \quad P(HC \cap PC) = 0.3.$$

Then,

$$P(HC \cup PC) = P(HC) + P(PC) - P(HC \cap PC) = 0.7 + 0.4 - 0.3 = 0.8$$

b) Has no computer;
Answer:

$$P(\overline{HC \cup PC}) = 1 - P(HC \cup PC) = 1 - 0.8 = 0.2$$

c) Has one and only one of the two types of computers.
Answer:

$$\begin{aligned} P((HC \setminus PC) \cup (PC \setminus HC)) &= P(HC \setminus PC) + P(PC \setminus HC) \\ &= P(HC) + P(PC) - 2P(HC \cap PC) = 0.5 \end{aligned}$$

11. The workers of a firm regularly use public transports (bus, metro, train) to go from home to work. It is known that:

- 54% use exclusively one of these public transports (bus-22%, metro-25%, train-7%);
- 44% use at least two of the three public transports.
- Additionally, 18% use bus and metro; 17% use bus and train; 19% use metro and train.

a) Define the events and represent them in a Venn diagram;
Answer:

$$B : \text{"A worker use the buses"}$$
$$M : \text{"A worker use the metro"}$$
$$T : \text{"A worker use the train"}.$$

b) Having in mind that there are other means of transport beyond the above mentioned, calculate the percentage of workers that don't use any of the above mentioned means of transport;

Answer: $P(\overline{A \cup B \cup C}) = 0.02$

c) Calculate the percentage of workers that use all the above mentioned means of transport.

Answer: 0.05

12. In a classroom there are 30 students, 20 boys and 10 girls. Four students are selected to form a committee representing the class.

- Calculate the probability that the first two selected are boys and the next two girls;

  Answer:

$$P(\text{"the first two selected are boys and the next two girls"}) = \frac{20 \times 19 \times 10 \times 9}{30 \times 29 \times 28 \times 27} = 0.052$$

- What is the probability that the committee has two girls and two boys?

  Answer:

$$P(\text{"the committee has two girls and two boys"}) = 3! \times \frac{20 \times 19 \times 10 \times 9}{30 \times 29 \times 28 \times 27} = 0.312$$

- What is the probability that the first student selected is a boy? And the third?

  Answer:

$$P(\text{"the first student selected is a boy"}) = \frac{20 \times 29 \times 28 \times 27}{30 \times 29 \times 28 \times 27} = \frac{2}{3}$$

$$P(\text{"the third student selected is a boy"}) = \frac{2}{3}$$

13. Consider a computer system that generates randomly a key-word for a new user composed of 5 letters (eventually repeated) of an alphabet of 26 letters (no distinction is made between capital and lower case letters). Calculate the probability that there is no repeated letters in the key-word.

Answer:

$$P(\text{"there is no repeated letters in the key-word"}) = \frac{{}^{26}A_5}{26^5} = 0.664$$

14. The weather forecast says that tomorrow it will rain with probability 0.4. Additionally, according to the weather forecast, the probability that:

- it will rain tomorrow and the day after is 0.2;
- it will rain tomorrow and it will be cloudy in the day after is 0.1.

a) What is the probability that it will rain the day after tomorrow given that it rains tomorrow?
Answer: $P(RA\,|RT) = 0.5$

b) What is the probability that it will be cloudy the day after tomorrow given that it rains tomorrow?
Answer: $P(CA\,|RT) = 0.25$

15. In a learning experience, an individual realizes twice consecutively a task where he can fail or be successful in any of them. The probability of failing the first attempt is 0.25. If he fails the first attempt the probability of being successful in the second is 0.5. If he is successful in the first attempt, the probability of failing in the second is 0.1. What is the probability of failing the second attempt?

Answer: Events:

$$T_1 = \text{"he fails the task in the first attempt"}$$
$$T_2 = \text{"he fails the task in the second attempt"}$$

Probabilities:

$$P(T_1) = 0.25, \quad P(\overline{T_2}|T_1) = 0.5, \quad P(T2|\overline{T_1}) = 0.1$$

Required Probability:

$$P(T_2) = P(T_2 \cap T_1) + P(T_2 \cap \overline{T_1}) = P(T_2|T_1) \times P(T_1) + P(T_2|\overline{T_1}) \times P(\overline{T_1})$$
$$= (1 - P(\overline{T_2}|T_1)) \times P(T_1) + P(T_2|\overline{T_1}) \times (1 - P(T_1)) = 0.2$$

16. A factory uses three machines to produce the same product. Machines A, B and C produce respectively 40%, 35% and 25% of total production. The percentage of defective parts produced by each machine are respectively 4%, 2% and 1%. If a piece is randomly selected from the total production.

a) What is the probability that it is not defective?
Answer: Events:

$$A = \text{"The product is produce by machine A"}$$
$$B = \text{"The product is produce by machine B"}$$
$$C = \text{"The product is produce by machine C"}$$
$$D = \text{"The product is defective"}$$

Probabilities:

$$P(A) = 0.4, \quad P(B) = 0.35, \quad P(C) = 0.25$$
$$P(D|A) = 0.04, \quad P(D|B) = 0.02, \quad P(D|C) = 0.01$$

Since the events A, B, C are a partition to the probability space

$$P(D) = P(D|A) \times P(A) + P(D|B) \times P(B) + P(D|C) \times P(C) = 0.0255$$

The required probability is $P(\overline{D}) = 1 - P(D) = 0.9745$.

b) Knowing that it is defective what is the probability that it has been produced by machine A?

Answer:

$$P(A|D) = \frac{P(D|A) \times P(A)}{P(D)} = 0.6275.$$

c) If two pieces are successively removed with replacement from the total production, what is the probability that one of them be defective and the other not?

Answer:

$$P(\text{"one of them be defective and the other not"}) = 2 \times P(D) \times P(\overline{D}) = 0.0497.$$

17. In a course of Statistics, students have to complete an exam in two hours. In the set of students that deliver the exam, 20% of them deliver it before the 2 hours, 50% of them deliver it on time and the remaining ones after the 2 hours. From the first ones, 70% have positive grade, from the second ones, 50% have positive grade, and, from the last ones 15% have positive grade.

a) What is the percentage of students that have positive grade?

Answer: Events:

BT ="The student delivers the exame before the 2 hours"

OT ="The student delivers the exame on time

AT ="The student delivers the exame after the 2 hours"

PG ="The student has a positive grade"

Probabilities:

$$P(BT) = 0.2, \quad P(OT) = 0.5, \quad P(AT) = 0.3$$
$$P(PG|BT) = 0.7, \quad P(PG|OT) = 0.5, \quad P(PG|AT) = 0.15$$

Required probability:

$$P(PG) = P(PG|BT) \times P(BT) + P(PG|OT) \times P(OT) + P(PG|AT) \times P(AT)$$
$$= 0.435.$$

b) Comment the following sentence: "in the set of students that have positive grade, more than half of them have delivered the exam on the regulatory time".

Answer:

$$\begin{aligned}P(BT \cup OT|PG) &= P(BT|PG) + P(OT|PG)\\ &= \frac{P(PG|BT)P(BT)}{P(PG)} + \frac{P(PG|OT)P(OT)}{P(PG)}\\ &= \frac{0.7 \times 0.2}{0.435} + \frac{0.5 \times 0.5}{0.435} = 0.8966\end{aligned}$$

Then we have a statement that is true!

c) Choose randomly 10 students that have delivered the exam. What is the probability that 4 of them have delivered it on time?

Answer:

$$\begin{aligned}P(\text{"4 of these 10 students have delivered on time"}) &= \binom{10}{4}(P(OT))^4(1-P(OT))^6\\ &= 0.2051\end{aligned}$$

18. Let A and B be two events in the sample space S, such that $P(A) \neq 0$ and $P(B) \neq 0$. Prove that if $P(A|B) \geq P(A)$ then $P(B|A) \geq P(B)$.

Answer: Since $P(A) \neq 0$ and $P(B) \neq 0$, then we have the following

$$P(B|A) = \frac{P(A|B) \times P(B)}{P(A)} \geq \frac{P(A) \times P(B)}{P(A)} = P(B).$$

19. At a production process, the produced items are tested for defects. A defective unit is classified as such with probability 0.9, whereas a correct unit is classified as such with probability 0.85. Furthermore, 10% of the produced units are defective.

a) Define the events and, by using these events, present all the probabilities mentioned in the description of the problem.

b) Compute (i) the probability that a unit is correct and (ii) the probability that a correct unit is classified as defective.

Answer: $P(\overline{D}) = 0.9$ and $P(\overline{D} \cap CD) = 0.153$.

c) Compute the conditional probability that a unit is defective, given that it has been classified as such.

Answer: $P(D|CD) = 0.4$.

20. Due to an economic downturn, a local factory is forced to lay off workers. The board of the factory has provided the following information:

- There are 1000 workers: 300 are managers and 700 are non-managers;
- The probability that a worker will be laid off is 0.25;
- The probability that a worker will be laid off given that he/she is a non-manager is 0.3.

a) Define the events and, by using these events, present all the probabilities mentioned in the description of the problem.

Answer:

Events:

$$L = \text{"A worker is laid off"};$$
$$M = \text{"A worker is a manager"}$$

$$P(M) = 0.3, \quad P(\overline{M}) = 0.7, \quad P(L) = 0.25, \quad P(L|\overline{M}) = 0.3$$

b) What is the probability that a worker will be laid off given that he/she is a manager.

Answer: By definition,

$$P(L|M) = \frac{P(L \cap M)}{0.3}.$$

Given that the events M and $\overline{M}$ are a partition for S, then

$$P(L) = P(L \cap M) + P(L \cap \overline{M}) \Leftrightarrow P(L) - P(L \cap \overline{M}) = P(L \cap M).$$

Since,

$$P(L \cap \overline{M}) = P(L|\overline{M}) \times P(\overline{M}) = 0.3 \times 0.7 = 0.21.$$

Therefore,

$$P(L|M) = \frac{P(L \cap M)}{0.3} = \frac{P(L) - P(L \cap \overline{M})}{0.3} = \frac{0.25 - 0.21}{0.3} = \frac{2}{15}.$$

c) According with the information provided by the board, compute the number of managers and non-managers that will be laid off and those that will not be laid off.

Answer:

21. An expert in soil perforation trusts that there is water in a certain region with probability 0.9. Additionally, from his experience, when there is water, in 75% of the time, water is not found in the first perforation.

	Manager	Non-Manager	Total
Layed-off	90	210	300
Non Payed-Off	210	490	700
Total	300	700	1000

a) What is the probability that water is found in the first perforation?
Answer:
Events:

$$W : \text{"There is water"}, \quad WF : \text{"Water was found in the first perfuration"}$$

and respective probabilities:

$$P(W) = 0.9, \quad P(\overline{WF}|W) = 0.75.$$

The probability that we want to compute is

$$\begin{aligned} P(WF) &= P(WF \cap W) + P(WF \cap \overline{W}) = P(WF \cap W) \\ &= P(WF|W)P(W) = (1 - P(\overline{WF}|W))P(W) = 0.225 \end{aligned}$$

b) In the event that water is not found in the first perforation, what is the probability that there is no water in that region?
Answer:

$$\begin{aligned} P(\overline{W}|\overline{WF}) = 1 - P(W|\overline{WF}) &= 1 - \frac{P(\overline{WF}|W) \times P(W)}{P(\overline{WF})} \\ &= 1 - \frac{0.75 \times 0.9}{1 - 0.225} \approx 0.12 \end{aligned}$$

22. Let A, B, C and D be four events in the sample space S, such that A, B and C are a partition of the sample space. Show that if

$$D \subset B \cup C \Rightarrow P(D) = P(\overline{A}) \times P(D|\overline{A}).$$

23. Let A and B be two independent events in the sample space S such that $P(A) = \frac{1}{3}$ and $P(B) = \frac{3}{4}$.

a) Compute $P(A \cup B)$, $P(A \cap B)$ and $P(B|A \cup B)$;
Answer:

$$P(A \cap B) = P(A) \times P(B) = \frac{1}{4}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{5}{6}$$

$$P(B|A \cup B) = \frac{P(B \cap (A \cup B))}{P(A \cup B)} = \frac{P(B)}{P(A \cup B)} = \frac{3/4}{5/6} = \frac{9}{10}$$

b) Show that A and $\overline{B}$ are independent events.

Answer:

$$\begin{aligned} P(A \cap \overline{B}) &= P(A) - P(A \cap B) = P(A) - P(A) \times P(B) \\ &= P(A)(1 - P(B)) = P(A) \times P(\overline{B}). \end{aligned}$$

24. Let A, B and C be three events in the sample space S, such that $P(A \cap B \cap C) = 0.1$, $P(A) = 0.5$, $P(B|A) = 0.4$. Compute the probability $P(C|A \cap B)$.

    Answer: $P(C|A \cap B) = \frac{1}{2}$.

25. Let A and B be two independents events in the sample space S. Supposing that $P(A \cup B) = 0.0298$, $P(A \cap B) = 0.0002$, $P(A) < P(B)$, find $P(A)$ and $P(B)$.

    Answer:

$$\begin{aligned} &0.0298 = P(A \cup B) = P(A) + P(B) - P(A \cap B) \Leftrightarrow 0.03 = P(A) + P(B) \\ &0.0002 = P(A \cap B) = P(A)P(B) \end{aligned}$$

From these equations, we get

$$0 = x^2 - 0.03x + 0.0002 \Leftrightarrow x = 0.01 \vee x = 0.02.$$

Then, $P(A) = 0.01$ e $P(B) = 0.02$.