ISEG - Lisbon School of Economics and Management

Statistics I

2^{nd} Semester of 2020/2021

Normal Assessment Exam

01 June 2021 | Duration: 2 hours

Name: Solution | Number: ____________

Please note the following:

- ☐ The multiple choice questions are marked with 1 value if the answer is correct and with -0.25 values if the answer is incorrect. If no answer is given, the question is marked with 0 values.
- ☐ In the remaining questions, justify your answers carefully and present all the calculations you consider necessary.

Question	1	2	3a	3b	4a	4b	5a	5b	6a	6b	7a	7b	8a	8b
Values	1	1	1	2	1	2	1	2	1	2	1	2	1	2
Assigned values														

1. Let A and B be two events of a sample space S such that A and B are independent and $P(B) = 2P(A)$. Prove that

$$P(A \cup B) = 3P(A) - 2(P(A))^2.$$

Compute $P(A \cup B)$ when $P(A \cap B) = 0.08$.

Answer 1.

A and B are independent when $P(A \cap B) = P(A)P(B)$

Therefore, $P(A \cap B) = 2(P(A))^2$

Since $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= P(A) + 2P(A) - 2(P(A))^2$

$= 3P(A) - 2(P(A))^2$

$0.08 = P(A \cap B) = 2(P(A))^2 \Leftrightarrow P(A) = \sqrt{0.04} = 0.2$

So, $P(A \cup B) = 3 \times 0.2 - 0.08$

$= 0.52$

PTO

2. Let X be an exponential random variable with parameter λ. Compute $E(e^{-X})$ and $Var(e^{-X})$.

Answer 2.

we know that $M_X(t) = (1 - t/\lambda)^{-1}$

So,

$$E(e^{-X}) = M_X(-1) = (1 + 1/\lambda)^{-1} = \frac{\lambda}{\lambda+1}$$

$$\begin{aligned} var(e^{-X}) &= E((e^{-X})^2) - (E(e^{-X}))^2 \\ &= E(e^{-2X}) - E(e^{-X})^2 \\ &= M_X(-2) - (M_X(-1))^2 \\ &= \frac{\lambda}{\lambda+2} - \left(\frac{\lambda}{\lambda+1}\right)^2 \end{aligned}$$

PTO

3. Assume that there are two boxes containing red and and green balls. Box 1 has 7 red balls and 3 green balls; Box 2 has 6 red balls and 2 green balls. A die is tossed. If the number of dots on the upper face of the die is less than 5, a ball is selected from the Box 1, otherwise the ball is selected from the Box 2.

a) Compute the probability that one gets a red ball and the number of dots on the upper face of the die is greater than 2.

☐ **i)** 29/40 ☐ **ii)** 43/60 ☐ **iii)** 21/40 ☒ **iv)** 29/60

b) Knowing that one gets a green ball, calculate the probability that this ball came from Box 2.

Answer 3.b)

RB : One gets a Red ball

GB : One gets a green ball

B_i : The ball was taken from Box i, $i = 1, 2$

$P(RB \mid B_1) = 7/10$ $P(RB \mid B_2) = 3/4$ $P(B_1) = 2/3$

$P(GB \mid B_1) = 3/10$ $P(GB \mid B_2) = 1/4$ $P(B_2) = 1/3$

Required probability:

$$P(B_2 \mid GB) = \frac{P(GB \mid B_2) \times P(B_2)}{P(GB)}$$

$$P(GB) = P(GB \mid B_1) \times P(B_1) + P(GB \mid B_2) \times P(B_2)$$

$$= 3/10 \times 2/3 + 1/4 \times 1/3 = 17/60$$

PTO

4. Let X be a random variable with a cumulative distribution function given by

$$F_X(x) = \begin{cases} 0, & x < a \\ x^3, & a \leq x < b \\ 1, & x \geq b \end{cases},$$

with $0 \leq a < b \leq 1$.

a) What is the correct statement?

☐ **i)** If $a = 0$ and $b = 1$, X is a mixed r.v.

☐ **ii)** If $b = 1$, X is always a continuous r.v.

☐ **iii)** If $a = \frac{1}{2}$, X is a continuous r.v.

☒ **iv)** If $a = \frac{1}{2}$, X is a mixed r.v.

b) Fix $a = 0$ and $b = 1$. Compute the probability density function, the expected value and the variance of X.

Answer 4.b)

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x^3, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

$$f_X(x) = F_X'(x) = \begin{cases} 3x^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$E(X) = \int_0^1 3x^3 \, dx = \frac{3}{4}$$

$$E(X^2) = \int_0^1 3x^4 \, dx = \frac{3}{5}$$

$$Var(X) = E(X^2) - (E(X))^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}$$

PTO

5. Let X_A and X_B represent the number of machines from brand A and B that are sold in one month. The marginal probability function of X_A and X_B is given by

$$f_{X_A}(x) = \begin{cases} 1/3, & x = 1 \\ 1/2, & x = 2 \\ 1/6, & x = 3 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_{X_B}(x) = \begin{cases} 1/4, & x = 1 \\ 1/4, & x = 2 \\ 1/2, & x = 3 \\ 0, & \text{otherwise} \end{cases}.$$

Assume that X_A and X_B are independent.

a) Calculate the probability that more than 4 machines are sold in a month.

☐ **i)** 1/2 ☒ **ii)** 3/8 ☐ **iii)** 1/12 ☐ **iv)** 5/8

b) Calculate the joint probability function of (X_A, X_B). You can present this function in a table, explaining how you get each entry. Compute the expected value of $X_A + X_B$.

Answer 5.b)

Since X_A and X_B are independent, then

$P(X_A = a, X_B = b) = P(X_A = a) \times P(X_B = b)$.

Thus,

b \ a	1	2	3	$P(X_B = b)$
1	1/12	1/12	1/6	1/3
2	1/8	1/8	1/4	1/2
3	1/24	1/24	1/12	1/6
$P(X_A = a)$	1/4	1/4	1/2	

$E(X_A + X_B) = E(X_A) + E(X_B) =$

$= 1 \times 1/4 + 2 \times 1/4 + 3 \times 1/2 + 1 \times 1/3 + 2 \times 1/2 + 3 \times 1/6$

PTO

$= 49/12$

6. Assume that the weekly profit of Companies A, B, C, and D, in thousand euros, is represented, respectively, by the random variables

$$X_A \sim N(5,4), \quad X_B \sim N(4,2), \quad X_C \sim N(4.5,6), \quad X_D \sim N(6,4).$$

Note that the normal distributions above are modeled with the mean and the variance ($N(\mu, \sigma^2)$). Assume independence between the random variables.

a) What is the company that has a greater probability of attaining a profit larger than 4.5 thousand euros in a week?

☐ **i)** Company A ☐ **ii)** Company B
☐ **iii)** Company C ☒ **iv)** Company D

b) Compute the probability that the profit of the four companies exceeds 18 thousand euros in a week.

Answer 6.b)

$$y = X_A + X_B + X_C + X_D \sim N(\mu, \sigma^2)$$

$$\mu = E(X_A + X_B + X_C + X_D) = 5+4+4.5+6 = 19.5$$

$$\sigma^2 = Var(X_A + X_B + X_C + X_D) \underset{\downarrow}{=} 4+2+6+4 = 16$$

due to the independency

$$Z = \frac{y - 19.5}{4} \sim N(0,1)$$

$$P(y > 18) = P\left(Z > \frac{18-19.5}{4}\right) = P(Z > -0.375)$$

$$\simeq P(Z > -0.38) = P(Z < 0.38)$$

$$= 0.6480$$

PTO

7. The length, in minutes, of a telephone call in a certain call center follows an exponential distribution with standard deviation 10 minutes.

In question a) use at least 3 decimal places in every intermediate computation.

a) The probability that, in 10 calls, the operator only spends more than 10 minutes in 3 calls is approximately

☒ **i)** 0.24 ☐ **ii)** 0.17 ☐ **iii)** 0.37 ☐ **iv)** 0.05

b) Assume that one operator answered 10 calls. Calculate the probability that shortest call has a duration longer than 1 minute. Explain your answer.

Answer 7.b)

If $X_1, X_2, \ldots, X_{10}$ represent the duration of each one of the 10 calls, then

$$Y = \min\{X_1, X_2, \ldots, X_{10}\}$$

represents the duration of the shortest call.

$Y \sim Exp(\tilde{\lambda})$ where $\tilde{\lambda} = 10 \times \lambda$

Since $Var(X_i) = \left(\frac{1}{\lambda}\right)^2$ and $\sigma(X) = 1/\lambda$

(because $X_i \sim Exp(\lambda)$) we can conclude that

$\lambda = \frac{1}{10}$ and $\tilde{\lambda} = 10 \times \frac{1}{10} = 1$

$$P(Y > 1) = e^{-\tilde{\lambda} \times 1} = e^{-1} \simeq 0.3679$$

PTO

8. Let X, Y and Z be three random variables such that $E(X) = 1$, $E(Y) = 3/2$, $E(Z) = -1$ and $Var(X) = 2$, $Var(Y) = 4$, $Var(Z) = 3$.

a) Assume that the random variables are all independent. Then, $Var(X - 2Y + 3Z)$ is

☐ **i)** 3 ☐ **ii)** 13 ☒ **iii)** 45 ☐ **iv)** 39

b) Assume now that $E(XY) = -1$, $E(XZ) = 2$ and $E(YZ) = 1$. Calculate $Cov(X + Y, Y + Z)$.

Answer 8.b)

$$Cov(X+Y, Y+Z) = Cov(X,Y) + Cov(X,Z) + Cov(Y,Y) + Cov(Y,Z)$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = -1 - 3/2 = -5/2$$

$$Cov(X,Z) = E(XZ) - E(X)E(Z) = 2 + 1 = 3$$

$$Cov(Y,Y) = Var(Y) = 4$$

$$Cov(Y,Z) = E(YZ) - E(Y)E(Z) = 1 + 3/2 = 5/2$$

$$Cov(X+Y, Y+Z) = -5/2 + 3 + 4 + 5/2 = 7$$

End

Formulae Sheet – Statistics I

- Probability:

Let A, B and C be events of a sample space. Then:

$$P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)$$

Let $\{A_1, A_2, \cdots\}$ be a partition of a sample space S and B an event of S. Then, if $P(A_j) > 0$, for all $j = 1, 2, \cdots$ and $P(B) > 0$,

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)P(A_j)}{\sum_{i=1} P(B|A_i)P(A_i)}.$$

- Expected values and other quantities (for a single RV):

Let X be a random variable. Then,

$$E(g(X)) = \int_{-\infty}^{+\infty} g(x) f_X(x) dx, \text{ if } X \text{ is a continuous RV}$$
$$E(g(X)) = \sum_{x \in D_X} g(x) f_X(x), \text{ if } X \text{ is a discrete RV}$$
$$Var(X) = E\left((X - \mu_X)^2\right) = E(X^2) - (E(X))^2$$
$$mo(X) = \arg\max_{x \in \mathbb{R}} f_X(x), \quad q_\alpha(X) = \min\{x \in \mathbb{R} : F_X(x) \geq \alpha\}, \quad M_X(t) = E(e^{Xt}).$$

- Expected values and other quantities (for functions of RV):

Let X and Y be two random variable. Then,

$$E(g(X,Y)) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x,y) f_{X,Y}(x,y) dx dy, \text{ if } X \text{ and } Y \text{ are continuous RV}$$
$$E(g(X,Y)) = \sum_{(x,y) \in D_{X,Y}} g(x,y) f_{X,Y}(x,y), \text{ if } X \text{ and } Y \text{ are discrete RV}$$
$$Cov(X,Y) = E\left((X - \mu_X)(Y - \mu_Y)\right) = E(XY) - E(X)E(Y)$$
$$E(g(X,Y)|Y = y) = \int_{-\infty}^{+\infty} g(x,y) f_{X|Y=y}(x) dx, \text{ if } X \text{ and } Y \text{ are continuous RV}$$
$$E(g(X,Y)|Y = y) = \sum_{x \in D_X} g(x,y) f_{X|Y=y}(x), \text{ if } X \text{ and } Y \text{ are discrete RV}$$
$$Var(X|Y = y) = E\left((X - \mu_{X|Y=y})^2 | Y = y\right) = E(X^2|Y = y) - (E(X|Y = y))^2.$$

- The tower property (or Law of total expectation):

Let X and Y be two random variable. Then,

$$E(E(X|Y)) = E(X) \quad \text{and} \quad E(E(Y|X)) = E(Y).$$

- Theoretical distributions:
  - Discrete uniform distribution:

$$f_X(x_j) = \frac{1}{k}, \; j = 1, 2, 3, ..., k$$
$$\text{if } D_X = \{1, 2, 3, \cdots, k\}, \text{ then } \mu_X = E(X) = \frac{k+1}{2}, \quad Var(X) = \frac{k^2 - 1}{12}$$
$$\text{if } D_X = \{0, 1, 2, 3, \cdots, m\}, \text{ then } \mu_X = E(X) = \frac{m}{2}, \quad Var(X) = \frac{m(m+2)}{12}$$

– Binomial (and Bernoulli) distribution ($X \sim Bin(n,p)$):

$$f_X(x) = \binom{n}{x} \times p^x(1-p)^{n-x},\ x = 0,1,2,\cdots,n$$
$$E(X) = np, \quad Var(X) = np(1-p) \quad M_X(t) = \left[(1-p) + pe^t\right]^n$$

– Hypergeometric distribution ($X \sim Hypergeometric(N,M,n)$):

$$P(X=k) = \frac{\binom{N-M}{n-k}\binom{M}{k}}{\binom{N}{n}}, \quad k = \max\{0, n-(N-M)\},\cdots,\min\{n,M\}$$
$$E(X) = n \times \frac{M}{N}, \quad Var(X) = n\frac{M}{N}\left(1-\frac{M}{N}\right)\frac{N-n}{N-1}$$

– Negative Binomial (and Geometric) distribution ($X \sim NB(k,p)$):

$$P(X=x) = \binom{x-1}{k-1}p^k(1-p)^{x-k}, \quad x = k, k+1, k+2, \cdots$$
$$E(X) = \frac{k}{p}, \quad Var(X) = \frac{k}{p}\left(\frac{1}{p}-1\right), \quad M_X(t) = \left(\frac{pe^t}{1-e^t(1-p)}\right)^k$$

– Poisson distribution ($X \sim Poi(\lambda)$):

$$P\left(X=x\right) = \frac{\lambda^x e^{-\lambda}}{x!}, x = 0,1,2,\cdots, \quad E(X) = Var(X) = \lambda, \quad M_X(X) = e^{\lambda\left(e^t-1\right)}$$

– Exponential distribution ($X \sim Exp(\lambda)$):

$$f_X(x) = \lambda e^{-\lambda x}, \quad x > 0, \quad E(X) = 1/\lambda, \quad Var(X) = 1/\lambda^2, \quad M_X(t) = (1-t/\lambda)^{-1}, t < \lambda.$$

– Gamma (and chi-squared) distribution ($X \sim Gamma(a,b)$):

$$E(X) = ab, \quad Var(X) = ab^2, \quad M_X(t) = (1-bt)^{-a}, t < 1/b$$
$$\text{If } a = 1,\ b = \frac{1}{\lambda} \text{ then } X \sim Exp(\lambda); \quad \text{If } a = \frac{n}{2},\ b = 2 \text{ then } X \sim \chi^2_{(n)}.$$

– Continuous uniform distribution ($X \sim U(a,b)$):

$$f_X(x) = \frac{1}{b-a}, \quad a < x < b, \quad E(X) = (a+b)/2, \quad Var(X) = (b-a)^2/12$$

– Normal distribution ($X \sim N(\mu,\sigma^2)$):

$$f_X\left(x\right) = \frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}, \quad E(X) = \mu, \quad Var(X) = \sigma^2, \quad M_X(t) = e^{\left(\mu t + 0.5\sigma^2 t^2\right)}$$

- Central Limit Theorem

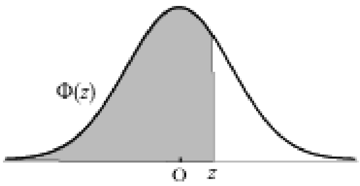
Let X_i be a sequence of independent and identically distributed random variables with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$. Then,

$$Z = \frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{n}\sigma} = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \overset{a}{\sim} N\left(0,1\right)$$

NORMAL DISTRIBUTION - Distribution function

TABELA 4 – DISTRIBUIÇÃO NORMAL – Função de distribuição

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

NORMAL DISTRIBUTION

TABELA 5 – DISTRIBUIÇÃO NORMAL: $\Phi^{-1}(z)$

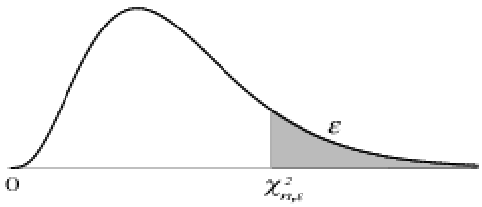
ε	.0005	.0010	.0050	.0100	.0200	.0250	.0500	.1000	.2000	.3000	.4000
z_{ε}	3.290	3.090	2.576	2.326	2.054	1.960	1.645	1.282	.842	.524	.253
$z_{\varepsilon/2}$	3.481	3.290	2.807	2.576	2.326	2.241	1.960	1.645	1.282	1.036	.842

$$z_{\varepsilon} : P(Z > z_{\varepsilon}) = \varepsilon \,; \quad z_{\varepsilon/2} : P(|Z| > z_{\varepsilon/2}) = \varepsilon \,.$$

CHI-SQUARE DISTRIBUTION

TABELA 6 – DISTRIBUIÇÃO DO QUI-QUADRADO

$$\chi^2_{n.\varepsilon} : P(X > \chi^2_{n.\varepsilon}) = \varepsilon$$



ε / n	.995	.990	.975	.950	.900	.750	.500	.250	.100	.050	.025	.010	.005	.001
1	.000	.000	.001	.004	.016	.102	.455	1.323	2.706	3.841	5.024	6.635	7.879	10.827
2	.010	.020	.051	.103	.211	.575	1.386	2.773	4.605	5.991	7.378	9.210	10.597	13.815
3	.072	.115	.216	.352	.584	1.213	2.366	4.108	6.251	7.815	9.348	11.345	12.838	16.266
4	.207	.297	.484	.711	1.064	1.923	3.357	5.385	7.779	9.488	11.143	13.277	14.860	18.466
5	.412	.554	.831	1.145	1.610	2.675	4.351	6.626	9.236	11.070	12.832	15.086	16.750	20.515
6	.676	.872	1.237	1.635	2.204	3.455	5.348	7.841	10.645	12.592	14.449	16.812	18.548	22.457
7	.989	1.239	1.690	2.167	2.833	4.255	6.346	9.037	12.017	14.067	16.013	18.475	20.278	24.321
8	1.344	1.647	2.180	2.733	3.490	5.071	7.344	10.219	13.362	15.507	17.535	20.090	21.955	26.124
9	1.735	2.088	2.700	3.325	4.168	5.899	8.343	11.389	14.684	16.919	19.023	21.666	23.589	27.877
10	2.156	2.558	3.247	3.940	4.865	6.737	9.342	12.549	15.987	18.307	20.483	23.209	25.188	29.588
11	2.603	3.053	3.816	4.575	5.578	7.584	10.341	13.701	17.275	19.675	21.920	24.725	26.757	31.264
12	3.074	3.571	4.404	5.226	6.304	8.438	11.340	14.845	18.549	21.026	23.337	26.217	28.300	32.909
13	3.565	4.107	5.009	5.892	7.041	9.299	12.340	15.984	19.812	22.362	24.736	27.688	29.819	34.527
14	4.075	4.660	5.629	6.571	7.790	10.165	13.339	17.117	21.064	23.685	26.119	29.141	31.319	36.124
15	4.601	5.229	6.262	7.261	8.547	11.037	14.339	18.245	22.307	24.996	27.488	30.578	32.801	37.698
16	5.142	5.812	6.908	7.962	9.312	11.912	15.338	19.369	23.542	26.296	28.845	32.000	34.267	39.252
17	5.697	6.408	7.564	8.672	10.085	12.792	16.338	20.489	24.769	27.587	30.191	33.409	35.718	40.791
18	6.265	7.015	8.231	9.390	10.865	13.675	17.338	21.605	25.989	28.869	31.526	34.805	37.156	42.312
19	6.844	7.633	8.907	10.117	11.651	14.562	18.338	22.718	27.204	30.144	32.852	36.191	38.582	43.819
20	7.434	8.260	9.591	10.851	12.443	15.452	19.337	23.828	28.412	31.410	34.170	37.566	39.997	45.314
21	8.034	8.897	10.283	11.591	13.240	16.344	20.337	24.935	29.615	32.671	35.479	38.932	41.401	46.796
22	8.643	9.542	10.982	12.338	14.041	17.240	21.337	26.039	30.813	33.924	36.781	40.289	42.796	48.268
23	9.260	10.196	11.689	13.091	14.848	18.137	22.337	27.141	32.007	35.172	38.076	41.638	44.181	49.728
24	9.886	10.856	12.401	13.848	15.659	19.037	23.337	28.241	33.196	36.415	39.364	42.980	45.558	51.179
25	10.520	11.524	13.120	14.611	16.473	19.939	24.337	29.339	34.382	37.652	40.646	44.314	46.928	52.619
26	11.160	12.198	13.844	15.379	17.292	20.843	25.336	30.435	35.563	38.885	41.923	45.642	48.290	54.051
27	11.808	12.878	14.573	16.151	18.114	21.749	26.336	31.528	36.741	40.113	43.195	46.963	49.645	55.475
28	12.461	13.565	15.308	16.928	18.939	22.657	27.336	32.620	37.916	41.337	44.461	48.278	50.994	56.892
29	13.121	14.256	16.047	17.708	19.768	23.567	28.336	33.711	39.087	42.557	45.722	49.588	52.335	58.301
30	13.787	14.953	16.791	18.493	20.599	24.478	29.336	34.800	40.256	43.773	46.979	50.892	53.672	59.702
40	20.707	22.164	24.433	26.509	29.051	33.660	39.335	45.616	51.805	55.758	59.342	63.691	66.766	73.403
50	27.991	29.707	32.357	34.764	37.689	42.942	49.335	56.334	63.167	67.505	71.420	76.154	79.490	86.660
60	35.534	37.485	40.482	43.188	46.459	52.294	59.335	66.981	74.397	79.082	83.298	88.379	91.952	99.608
70	43.275	45.442	48.758	51.739	55.329	61.698	69.334	77.577	85.527	90.531	95.023	100.425	104.215	112.317
80	51.172	53.540	57.153	60.391	64.278	71.145	79.334	88.130	96.578	101.879	106.629	112.329	116.321	124.839
90	59.196	61.754	65.647	69.126	73.291	80.625	89.334	98.650	107.565	113.145	118.136	124.116	128.299	137.208
100	67.328	70.065	74.222	77.929	82.358	90.133	99.334	109.141	118.498	124.342	129.561	135.807	140.170	149.449